DESCRIPTIVE GEOMETRY FOR ENGINEERS

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Descriptive Geometry for Engineers

DESCRIPTIVE GEOMETRY FOR ENGINEERS

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PREFACE

The limited amount of time that can be devoted to this important subject in engineering schools prompted the authors to develop to a high degree of utility the methods that are most essential in making drawings.

This book presents the necessary principles involved in the solution of any engineering problem, they are presented in a logical order and are applied to practical problems.

The special feature of the book, Chapter 10, "Hip and Valley Angles for Roofs of Buildings," is the connecting link of the fundamentals in the application to practical problems. By the simple process of turning a roof upside down one may solve by the same principles such problems as hoppers, bins, chutes, transition pieces, air conduits, Imhof tanks, intake chambers for water turbines, steam turbines, intake and exhaust manifold for various purposes. As all engineers are concerned with the design of these structures, the principles are developed in detail.

The graphical solution of the ten principal angles and bevels are solved by the Successive Projection and Revolution Methods.

It is the surviving author's purpose to explain in another volume the details for the cutting of beams and plates, as well as the bending of the plates and punching of holes in the design of various skewed connections.

Many problems may be solved more economically by the Method of Traces given in Chapter 9. Some young instructors are at variance on this point, but the author's experience prompts him to introduce this method for those who know the economic value of this solution.

The third angle of projection is used, which is practically universal throughout this country.

Students and instructors are urged to concentrate on and thoroughly understand the sentences printed in *italics*.

The text is supplemented by a separate loose leaf book of plates ($8\frac{1}{2} \times 11$), entitled PROBLEMS IN DESCRIPTIVE GEOMETRY FOR ENGINEERS in which the problems are to be solved by both the graphical and mathematical methods.

The surviving author wishes to express his grateful appreciation to Professor Harry Bradley who wrote the first manual presenting this method. Gratitude is expressed to Professor Hale Sutherland, Head of the Department of Civil Engineering at Lehigh University, without whose interest, kindly help and cooperation this book would not have been written.

Many thanks are due my colleagues and friends for valuable advice and assistance.

E. H. U.

Bethlehem, Pa.
January, 1937.

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PROBLEMS BY THE TRACE OF PLANE METHOD

Chapter 1

Orthographic Projection

1. *Descriptive Geometry* deals with the theory that underlies a system of drawing known as orthographic projection.

In engineering and construction work the use of orthographic projections, or views, as they are called, is universal. These views are not concerned primarily with the actual appearance of the object, however important that may be from an aesthetic or commercial point of view, but are for the purpose of showing its true shape and form, and the actual sizes and relative positions of its various parts.

An orthographic projection is based on the properties of a right angle, that is, on the properties of lines and planes which are mutually perpendicular.

2. *Fundamental Conceptions.* Through any point in space three mutually perpendicular lines can be drawn. Taking the point as origin, any other point in space can be located by not more than three coordinates parallel to these lines. In this sense, space may be said to be three-dimensioned.

Through any point in a plane, two mutually perpendicular lines can be drawn. Taking the point as origin, any other point in the plane can be located by not more than two coordinates parallel to these lines. In this sense, a plane may be said to be two-dimensioned.

The problem solved by orthographic projection is to define completely actual objects, occupying three-dimensional space, by means of views (projections) drawn on a two-dimensional plane.

3. *Planes of Projection.* Starting with any set of three mutually perpendicular lines in space as axes, pass a plane parallel to any two of them. This plane will be perpendicular to the third axis, and the remaining two axes may be projected orthographically on the plane in a direction parallel to the third axis. Applying this process to a solid object in space, the result is a projection, or view, in which two of the rectangular dimensions of the object appear in their true relative positions and lengths, while the third dimension is entirely suppressed. We have thus succeeded in forcing the two dimensions of a plane to represent two of the three dimensions of the solid object.

Obviously, however, a single orthographic view does not completely represent a solid. Pass a second plane parallel to the third axis, and to either of the others, say the first. This plane is perpendicular to the second axis, and the first and third axes may be projected orthographically on the plane in a direction parallel to the second axis. Applying to a solid, the result is again a view showing two dimensions, one of which is the dimension which failed to appear in the view first made.

It follows that at least two orthographic projections are necessary to represent or locate objects—by which term is meant points, lines, and surfaces, as well as solids—situated in ordinary, three-dimensional space. Also, it is evident that the two planes of projection used are at right angles to each other, and that each is parallel to the axis, or dimension, which appears in both views.

4. *Choice of the Planes of Projection.* The stability of any construction made at any particular place on the earth's surface depends on the relation of its parts to the direction of the force of gravity at that place. This direction is known as the vertical, and furnishes a natural direction for a plane of projection. A plane at right angles to the vertical is horizontal, or level, which is also a natural direction. When but two projections of an object are made,

therefore, the planes usually chosen are, one horizontal, the other vertical. Indeed, so fundamental are these planes that the projections made on them have special names. *A view drawn on a horizontal plane is a plan; a view drawn on a vertical plane is an elevation.*

Note that the horizontal direction is determined by a plane, so that all horizontal planes are parallel. But the vertical direction is determined by a line (the plumb line), so that any two vertical planes are not necessarily parallel, and the choice of one as a plane of projection is to some extent arbitrary. In fact, two vertical planes may be at right angles to each other, so that it is possible for an object to be represented by two elevations.

5. *Point of View.* A projection is always considered as a view of the object as seen from a particular direction. To be sure, natural vision, with an eye, or eyes, occupying a fixed station, cannot see the object in the form in which it is projected. The imaginary vision which sees the object must be able to look simultaneously at every point of the object in a direction perpendicular to the plane on which the view is drawn. Nevertheless, such a form of artificial vision is not difficult to conceive, and the direction of the resulting lines of sight is of primary importance.

A plan is a view of an object as seen from above. The direction of sight is vertical, and downward.

An elevation is a view of an object as seen from in front. The direction of sight is horizontal, and from front to back.

These two directions of sight are fundamental, and invariable.

6. *Arrangement of Views.* In Fig. 1 are given the plan and elevation of a regular five-sided prism, standing upright on one of its bases. The pentagon is the *plan,* or view of the prism as seen from above. The other view is the *elevation,* or view from in front. The two views are lined

up in a vertical direction, so that the distances which appear in each view, such as the width from *c* to *d*, or from *b* to *e*, can be projected from one view to the other. In this figure, the plan is placed above the elevation.

In Fig. 2 are shown the same plan and elevation, lined up in a vertical direction as before, but with the plan placed below the elevation.

In Figs. 3 and 4 are shown plan and elevation of the frustum of a right circular cone. In either figure, the views show the object to be standing upright and resting on its larger base; but in Fig. 3 the plan is above the elevation, and in Fig. 4 the plan is below the elevation.

Which is the better arrangement, that of Figs. 1 and 3, or that of Figs. 2 and 4? Unfortunately, a final and definite answer to this question, to which no one will take exception, cannot be given. Before we make a choice, let us study the possibilities of the two arrangements.

7. *A Study of Arrangement.* See Figs. 5, 6, and 7. A square pyramid is taken, and placed in four different positions, two with the base in a horizontal plane, and two with the base in a vertical plane. These positions are shown pictorially in Fig. 5. The corresponding projections are shown in Fig. 6, with the plan placed above the elevation, and in Fig. 7 with the plan placed below the elevation. By comparison of Figs. 6 and 7 it will be seen that for each position of the pyramid the views are identical, plan with plan and elevation with elevation, and differ only in arrangement.

8. *Visibility.* In the projections shown in Figs. 1, 2, 6, and 7, some of the lines are drawn dotted, indicating hidden edges of the solid. *These invisible lines represent the lower edges if drawn in a plan, and the back edges if drawn in an elevation.* As has just been noted, the views are identical, whether the plan is above or below the elevation, so that the determination of visible and invisible lines

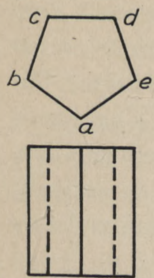

Fig. 1.

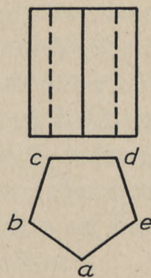

Fig. 2.

Fig. 3.

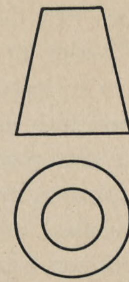

Fig. 4.

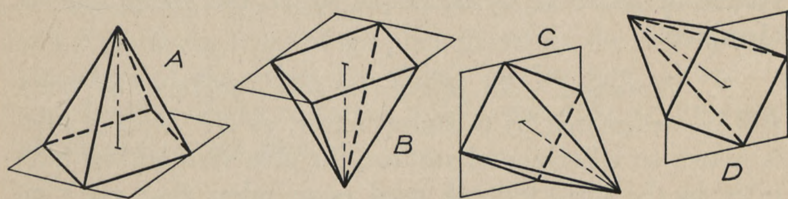

Fig. 5.

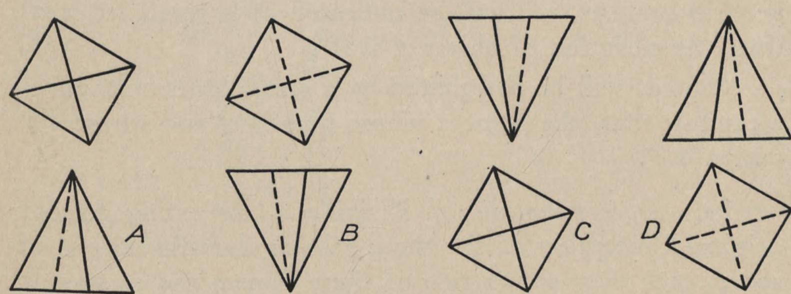

Fig. 6.

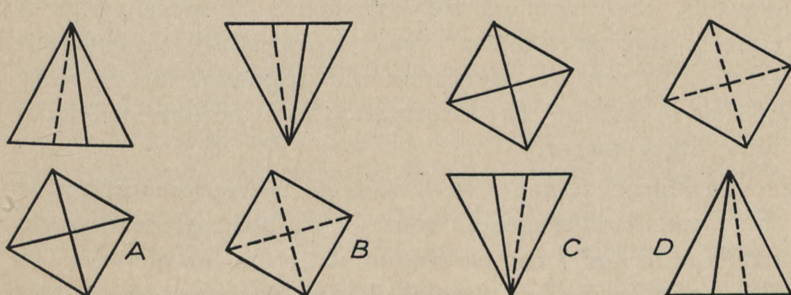

Fig. 7

does not depend on the arrangement of the views. But, in order to determine the visibility, it is absolutely necessary to know which view is plan and which is elevation. To illustrate, compare *A*, Fig. 6, with *D*, Fig. 7. The views are the same in form and arrangement, but the visibility, as shown by the interior lines in each projection, is entirely reversed. The same may be noted by other comparisons between Figs. 6 and 7.

We must therefore, for present purposes, adopt a system of notation, by means of which the views may be identified.

9. *Notation.* An object (a solid, surface, or line) will be designated by a capital letter, as *A*. Since a plan is drawn on a horizontal surface, the plan view will be indicated by the small letter *h,* used as an index; thus, the plan of object *A* is A^h. Similarly, the elevation is drawn on a vertical surface, and will be indicated by a small letter *v*; thus, the elevation of object *A* is A^v.

A point will be designated by a small letter instead of a capital; thus, the point *c*, whose plan is c^h and whose elevation is c^v.

10. *Determination of Visibility.* Reverting to our original conception of the three rectangular dimensions of space, each view shows two of these dimensions as seen in the direction of the third. The visibility of any view depends, therefore, on the coordinate of space which is entirely suppressed in that view. It follows that no amount of study of any one projection will determine the visibility of that projection; the information must be obtained from some other source.

In Figs. 5, 6, and 7 we have pictorial representations of the objects, and the visibility as shown in the plans throughout Figs. 6 and 7 can be determined from the pictures. In Fig. 1 the position of the object was stated in words. But in general an object is determined by projections alone, so

that we may say that the visibility shown in any projection must be determined by information obtained from some other projection.

Examples of Visibility. In each example, the views are given at the left in a fine continuous line, as they would be made in a pencil drawing before the visibility is determined. At the right the same views are shown after the visibility has been determined.

Example 1, Fig. 8. A six-sided prism. The notation N^h shows that the hexagon is the plan. The points b^h and c^h are at the front of the plan, therefore the corresponding edges of the prism are visible in elevation. *That is, the visibility of the elevation is determined by information obtained from the plan.*

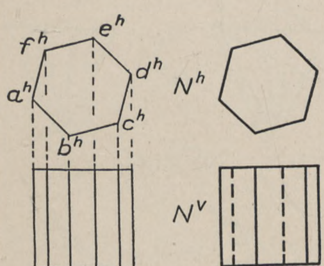

Fig. 8.

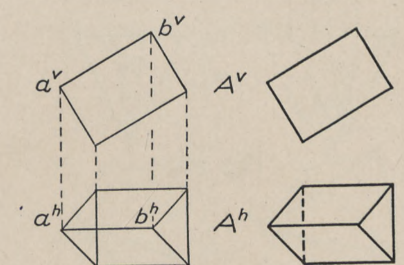

Fig. 9.

Example 2, Fig. 9. A triangular right prism in an inclined position. The notation A^v shows that the rectangle is the elevation. The point b^v is the highest corner of the elevation, and therefore represents the highest corner of the prism. Hence in the plan the three edges meeting at b^h are visible. *That is, the visibility of the plan is determined by information obtained from the elevation.*

Example 3, Fig. 10. A square pyramid. The notation shows the square is the plan. The elevation shows that the vertex, o, is the lowest point of the pyramid, therefore the interior lines are invisible in plan. The plan shows

that the point a is the front corner of the base, therefore the corresponding lateral edge is visible in elevation. *Here, as is usually the case, the visibility of each view is determined by information obtained from the other view.* Compare B, Fig. 6.

Example 4, Fig. 11. A square pyramid. The square is the elevation. The plan shows that the vertex, o, is the front point of the pyramid, therefore the interior lines are visible in elevation. The elevation shows that the point e is the highest corner of the base, therefore the corresponding lateral edge is visible in plan. Compare C, Fig. 7.

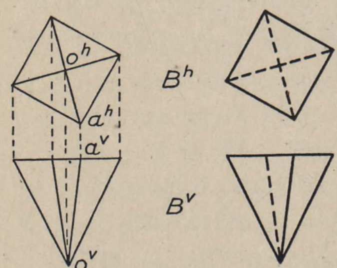

Fig. 10.

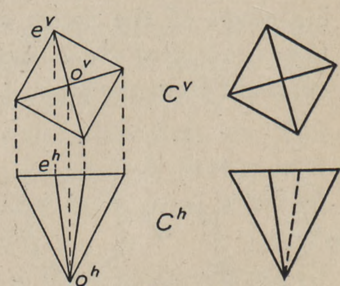

Fig. 11.

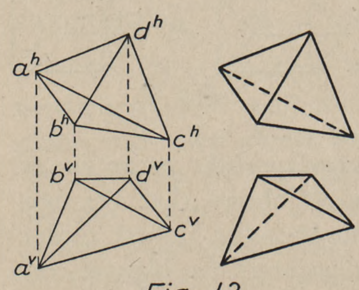

Fig. 12.

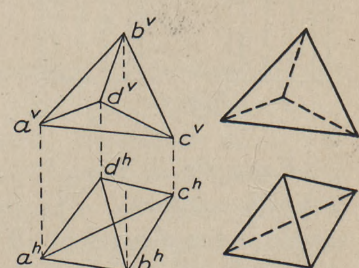

Fig. 13.

Example 5, Fig. 12. A tetrahedron, or solid bounded by four triangular faces. *The highest line in the elevation is $b^v d^v$; therefore $b^h d^h$ is visible in plan. The front line of the plan is $b^h c^h$; therefore $b^v c^v$ is visible in elevation.*

Example 6, Fig. 13. A tetrahedron. Point b^v is the highest point of the elevation, but this fact does not deter-

mine the visibility of the plan. Line $a^v c^v$ is the lowest line of the elevation, therefore $a^h c^h$ is invisible in plan. Point b^h is the front point of the plan, but this does not determine the visibility of the elevation. Point d^h is the back point of the plan, therefore in the elevation the three edges meeting at d^v are invisible. *Here it is the invisible lines, and not the visible ones, which are directly determined.*

Example 7, Fig. 14. A tetrahedron. The highest line in the elevation is $b^v c^v$, but as $b^h c^h$ is on the outline of the plan, this fact does not determine the visibility. The lowest line in the elevation is $a^v d^v$; but as $a^h d^h$ is also on the outline of the plan, the visibility is still undetermined. The same two edges of the solid, namely ac and bd, are interior lines in each view.

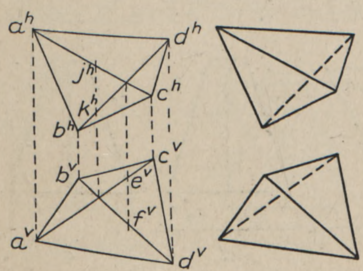

Fig. 14.

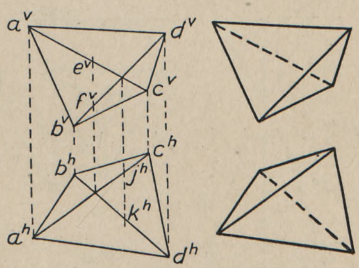

Fig. 15.

Project the intersection of $a^h c^h$ and $b^h d^h$ to the elevation, obtaining point e^v on $a^v c^v$ and f^v on $b^v d^v$. Point e^v is above point f^v, and point e^v lies on $a^v c^v$; therefore $a^h c^h$ is visible in plan.

To determine the visibility of the elevation, project the intersection of $a^v c^v$ and $b^v d^v$ to the plan. Point k^h on $b^h d^h$ is in front of point j^h on $a^h c^h$; therefore line $b^v d^v$ is visible in elevation.

Example 8, Fig. 15. A tetrahedron. The same difficulties arise as in Fig. 14. To determine the visibility of the plan, project the intersection of $a^h c^h$ and $b^h d^h$ to the eleva-

tion. Point e^v on a^vc^v is above point f^v on b^vd^v; therefore a^hc^h is visible in plan. To determine the visibility of the elevation, project the intersection of a^vc^v and b^vd^v to the plan. Point k^h on b^hd^h is in front of j^h *on* a^hc^h; therefore b^vd^v is visible in elevation.

11. *Location of Planes of Projection.* So long as only a plan and an elevation of an object are made, it is sufficient to know that a plan is drawn on a horizontal plane, and an elevation on a vertical plane, without giving any fixed or definite location to these planes. But two views of an object are not always enough; and before any additional views can be constructed, one or both of the planes on which the original views are made must be given a fixed position with respect to the object.

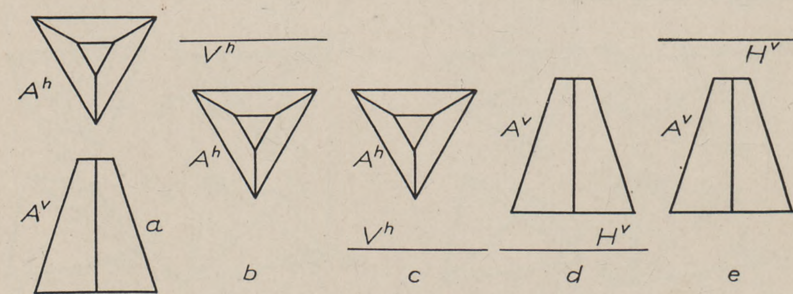

Fig. 16.

Let us take for an object the frustum of a triangular pyramid, placed as shown by the plan and elevation at *a*, Fig. 16. Let it be supposed that the plan is drawn on a horizontal plane H, and the elevation on a vertical plane V.

Consider first the plan. *Wherever V is located, it will appear in edge view in the plan as a horizontal straight line.* If V is behind the object, it will show behind the plan, as V^h, Fig. 16, *b*. If V is in front of the object, V^h is in front of the plan, as in Fig. 16, *c*.

Consider the elevation. *In the elevation H will appear in edge view as a horizontal straight line, H^v.* If H is

below the object, H^v is below the elevation, as in Fig. 16, *d*. If H is above the object, H^v is above the elevation, as in Fig. 16, *e*.

The planes H and V intersect in a horizontal straight line. It is convenient to represent that line of intersection by drawing a horizontal straight line across the paper. Having done so, select from Fig. 16 one of the plans at *b* or *c* and one of the elevations at *d* or *e*, and line them up vertically, making both V^h and H^v coincide with the horizontal line just drawn. Four combinations are possible, which are shown in Fig. 17. Two of the arrangements, except for the addition of the horizontal line V^hH^v, are readily recognized. The other two, namely, the second and fourth, are perhaps somewhat startling. All the arrangements can be accounted for, however, by assuming the planes H and V to be indefinite in extent.

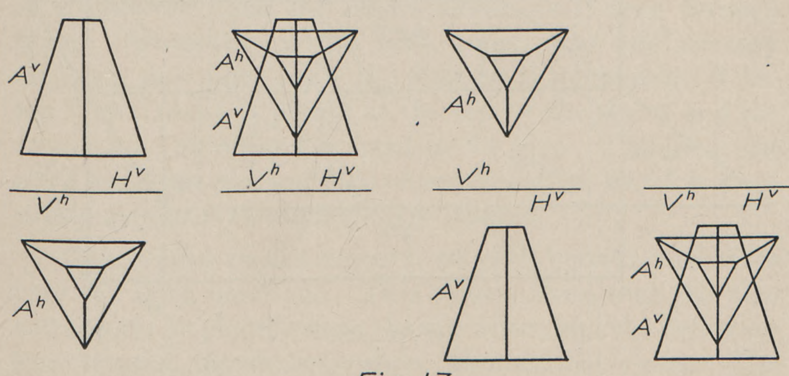

Fig. 17.

12. *The Four Quadrants.* See Fig. 18. Let H and V be imagined indefinite in extent, and to intersect in the line RL. These planes divide all space into four quarters, or quadrants, which have been arbitrarily designated as first, second, third, and fourth, as shown by the numbers at the right of the figure. Place a point in each quadrant in turn, and project on H and V. The result is pictured in Fig. 18,

and the corresponding placing of plan and elevation is that of Fig. 17, where RL is represented by the horizontal line H^vV^h. That is, in Fig. 17 the object is placed in the first, second, third, and fourth quadrants respectively.

13. *The Reference Line.* The line of intersection of H and V, lettered RL in Fig. 18, and H^vV^h in Fig. 17, will hereafter be called the reference line. It will be seen that the reference line is between the views if the object is in the first or third quadrants, below both views if the object is in the second quadrant, and above both views if the object is in the fourth quadrant.

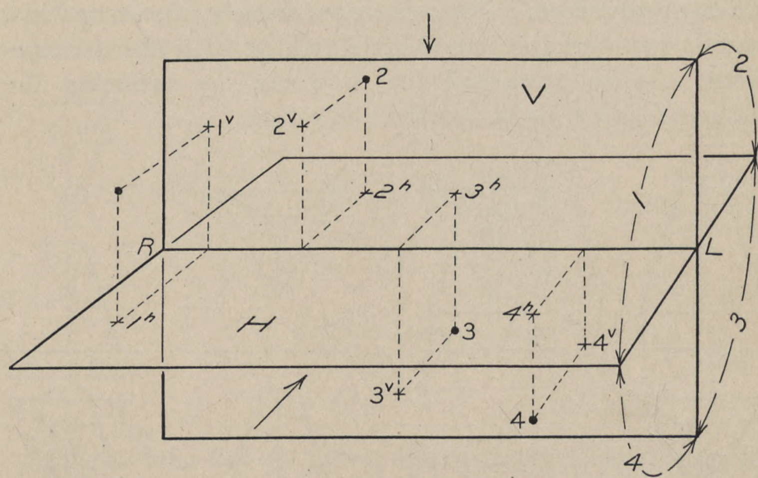

Fig. 18.

14. *Choice of Quadrant.* We are now ready to make a choice of the quadrant in which an object shall ordinarily be drawn; or, in other words, a choice of the relative positions of the views and the reference line.

The second and fourth quadrants we may reject at once. To be sure, the result need not be so bad as shown in Fig. 17. By carefully choosing the relative distances of H and V from the object, the views need not overlap; but the same

result can be more easily and surely obtained by using the first or third quadrant.

The first quadrant has an historic background. Gaspard Monge (1746-1818), a Frenchman, issued in 1799 the first text on Descriptive Geometry. He drew a reference line horizontally across the middle of the sheet, placed the plan in front of this line, and the elevation above it. This is the first quadrant method, plan below elevation. In this placing, it can be imagined that the planes of projection are opaque, and that light, coming from behind and above the observer, can cast a shadow of the object on them. Possibly this was the reason for the original choice of this quadrant, as shades and shadows play an important part in architectural design.

In the design of machinery, however, shadows have no part; nor is it difficult to imagine the planes of projection as transparent, and the object drawn as if seen through them. Moreover, it has been found that the average mechanic sees more easily the piece he is to construct if the top view (plan) is placed at the top of the sheet, and the front view (elevation) in front of the plan. *This gives the third quadrant placing, plan above elevation, which is practically universal throughout this country.*

Conforming to this practice, we shall hereafter consider only the third quadrant. This means, then, that the plan will always be placed above the elevation, and the reference line, if used, drawn somewhere between the views. The planes of projection are transparent, and each projection is a view of the object as seen by looking through the plane to the object beyond.

15. *Additional Views. If we are given the plan and elevation of an object, or in general any two projections on two planes at right angles to each other, we may construct as many additional views of the object as may be needed or desired.*

In Fig. 19 are shown the plan and elevation of an object, together with a number of additional projections from various points of view. The object chosen is a tetrahedron, because this solid has but four vertices, so that there are but four points to follow throughout the various projections.

The theory which underlies the making of the additional views will be discussed in full in the next chapter. At this point we give only the method of their construction.

Before making any of the additional views, the horizontal reference line *RL1* is drawn between the plan and elevation, thus locating the position of *H* and *V*.

16. *The Profile View.* The additional view most frequently made in practice is a profile view, usually called a *side elevation.*

The profile plane of projection is a plane which is at right angles to both *H* and *V*. It will be designated by the capital letter *P*, and projections on it by the small letter *p*.

Since the profile plane is perpendicular to both *H* and *V*, it is perpendicular to their line of intersection *RL1*. Hence draw the vertical line *RL2*, at the right or left of the elevation, according to whether a right or left side view is wanted. The distance of *RL2* from the elevation is arbitrary; but wherever *RL2* is drawn, it represents the intersection of *V* and *P*.

Project from a^v, b^v, c^v, and d^v, perpendicular to *RL2*. If *RL1* is read as a part of the plan, it represents an edge view of *V*, and shows the distance of *V* in front of the object. If *RL2* is read as a part of the profile projection, it also represents an edge view of *V*, and also shows the distance of *V* in front of the object. Hence the distance from *RL2* to a^p equals the distance from *RL1* to a^h; the distance *RL2* to b^p equals the distance *RL1* to b^h, and so on. The particular profile view shown in Fig. 19 is the *right side elevation.*

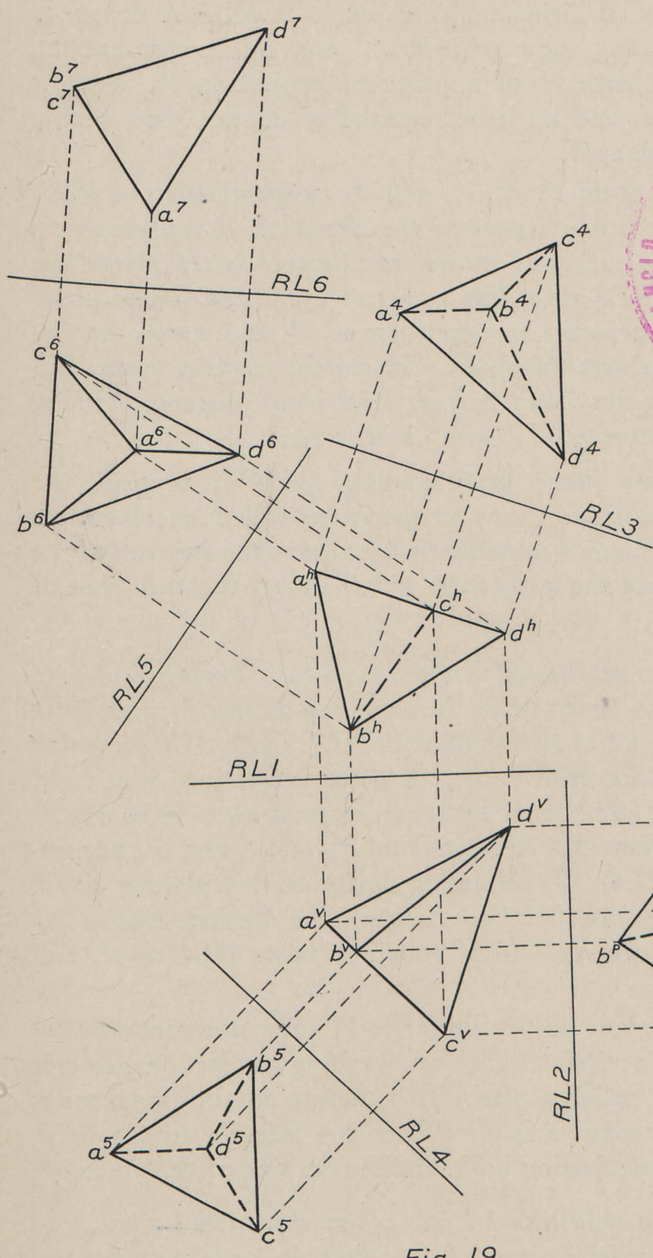
d7
b7
c7
a7
RL6
c4
a4
b4
c6
a6
d6
d4
RL3
RL5
ah
ch
dh
bh
RL1
dv
dP
av
aP
bv
bP
cv
cP
b5
RL2
a5
d5
RL4
c5

Fig. 19.

17. *An Additional Elevation.* Let *RL3* be drawn in the plan at any angle with *RL1.* Let *RL3* represent the intersection with H of a plane of projection, 4, at right angles to H, and let it be required to draw a view of the object on plane 4.

Project from a^h, b^h, c^h, and d^h, perpendicular to *RL3.* If *RL1* is read as a part of the elevation, it represents an edge view of H, and shows the distance of H above the object. If *RL3* is read as a part of the projection on plane 4, it also represents an edge view of H, and shows the distance of H above the object. Hence the distance from *RL3* to a^4 equals the distance from *RL1* to a^v, distance *RL3* to b^4 equals distance *RL1* to b^v, and so on.

Since any plane perpendicular to H is vertical, the resultant projection may properly be called an elevation. In this case plane 4 is evidently parallel to the face *acd* of the solid, so that the projection $a^4c^4d^4$ shows the true size of this face.

18. *An additional View on a Plane Perpendicular to V.* Let *RL4* be drawn in the elevation at any distance from the object, and at any angle with *RL1.* Let *RL4* represent the intersection with V of a plane of projection, *5*, at right angles to V, and let it be required to draw a view of the object on plane *5*. Project from a^v, b^v, c^v, and d^v, perpendicular to *RL4*. By reasoning similar to that already given, the distance from *RL4* to a^5 equals the distance from *RL1* to a^h; distance *RL4* to b^5 equals distance *RL1* to b^h, and so on.

Since a plane perpendicular to V is not horizontal unless it is also parallel to H, the resulting projection can not properly be called a plan. It is merely an additional view. In this case plane 5 is parallel to the face *abc* of the solid, so that the projection $a^5b^5c^5$ is the true size of that face.

19. *An Additional View of an Additional View.* In making additional views of an object, it is not necessary to

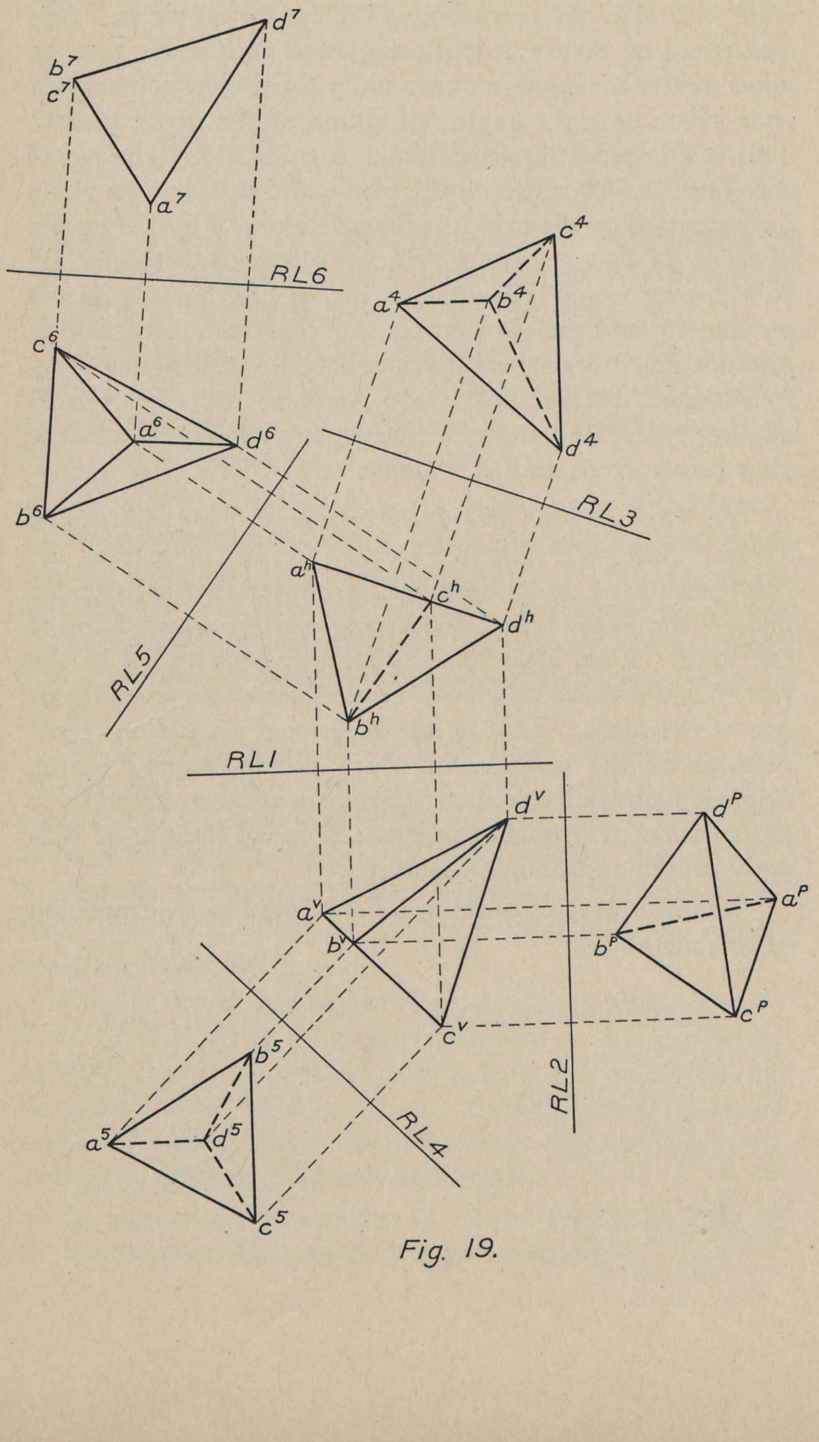
d^7
b^7
c^7
a^7
RL6
c^4
a^4
b^4
c^6
a^6
d^6
d^4
b^6
RL3
a^h
c^h
d^h
RL5
b^h
RL1
d^v
d^P
a^v
a^P
b^v
b^P
c^v
c^P
b^5
RL2
a^5
d^5
RL4
c^5

Fig. 19.

start with plan and elevation as the two given views. Any two views on planes at right angles to each other may be taken as the basic projections, and a third view constructed on a plane at right angles to either of the given planes. This is shown by the views 6 and 7 in Fig. 19. The reference line *RL5* is drawn on the plane, and the view on plane 6 constructed on the principle that distance *RL5* to a^6 equals distance *RL1* to a^v, distance *RL5* to b^6 equals distance *RL1* to b^v, and so on. Then the reference line *RL6* is drawn on plane 6, and the view on plane 7 constructed by making distance *RL6* to a^7 equal distance *RL5* to a^h, distance *RL6* to b^7 equal distance *RL5* to b^h, and so on. If desired, another reference line can be drawn on plane 7 and another view constructed; and thus indefinitely.

20. *Solution of Problems by the Use of Additional Views.* Still referring to Fig. 19, we may note that the reference lines 5 and 6 have been so chosen that the view on plane 7 shows the true dihedral angle between two faces of the solid. Also, attention has already been called to the fact that the views on planes 4 and 5 show the true size of one of the faces. That is, the additional projections show features of the given solid which do not appear in the original plan and elevation. This suggests a method for the solution of problems in space, namely, the use of properly chosen additional projections.

The following chapters will be devoted to an exposition of this method.

Chapter 2

Fundamental Constructions

21. *Number of Planes Required.* Any problem in descriptive geometry can be solved wholly by the use of a sufficient number of suitably chosen planes.

22. *Number of Views Required.* It has already been shown that at least two views are required to represent space on a flat surface. *So that two views, locating the data, are the minimum in any problem.*

It will be shown in the present chapter that the fundamental constructions on the point, line, and plane never require more than two additional views, or four in all, no matter how the data are located. Problems which are based directly on one of these fundamental constructions will therefore never require more than four views in all to solve.

Problems involving several steps may, however, exceed this number. There are a few problems which require a total of five views to solve under the most unfavorable positions of the data, and one or two others which, theoretically solvable in four views, are much more readily solvable in five. *So that, in general, five views may be taken as the practical maximum.*

On the other hand, there are problems which require three or four views when solved wholly by the method of successive projection, which can be solved in two views by the adoption of other methods. Some of these methods will be developed later; they are best deferred until the method of projection is well understood.

In any event, the number of views should be kept to the minimum consistent with the method used.

23. *Projection of a Point on Two Planes.* Let two planes of reference, A and B, intersect in a line L (Figs. 20, 21, 22). Let o be a point in space, and let o be projected orthographically on the two planes at o^1 and o^2. Let the two planes be developed, by unfolding about the line L, into a single plane, as shown. Then the line $o^1 \cdot o^2$, connecting the two projections, is perpendicular to the line L.

Proof. Since $o \cdot o^1$ is perpendicular to A, and $o \cdot o^2$ is perpendicular to B, the plane of these two lines is perpendicular to both A and B, and consequently is perpendicular to their line of intersection, L. Let the plane of $o \cdot o^1$ and $o \cdot o^2$ intersect L at point x; then the lines of intersection, $o^1 \cdot x$ and $o^2 \cdot x$, are both perpendicular to L. Hence, after development, $o^1 \cdot x$ and $o^2 \cdot x$ are both perpendicular to L at the same point, and so form one continuous line perpendicular to L.

Conversely, if two projections of a point are given on a single plane, and it is known that the planes of reference have been developed as above, the direction (but not necessarily the location) of the line L becomes known, since L is perpendicular to the line joining the two given projections. Hence it is that if two views, such as plan and elevation, or front and side elevation, are given "lined up," as is usual, planes of reference may be assumed by drawing a reference line which connects the two views.

But although the lines $o^1 \cdot o^2$ and L are mutually perpendicular, we should note particularly that this relation does not depend in any way upon the angle between the planes A and B. As shown in Figs. 20, 21, and 22, this angle may be acute, right, or obtuse.

We should further note that if the planes A and B are not at right angles to each other, as in Figs. 20 and 22, the distances $o^1 \cdot x$ and $o^2 \cdot x$, from the projections to the line L, do not equal the distances in space from the point to the planes A and B. To show these distances in the drawing, the planes A and B must be at right angles, as in Fig. 21.

Then the figure $o{\cdot}o^1{\cdot}x{\cdot}o^2{\cdot}o$ becomes a rectangle. Hence $o^2{\cdot}x = o{\cdot}o^1 =$ distance of point o from A, and $o^1{\cdot}x = o{\cdot}o^2 =$ distance of point o from B.

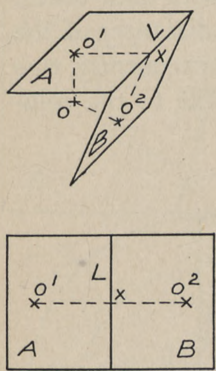

Fig. 20.

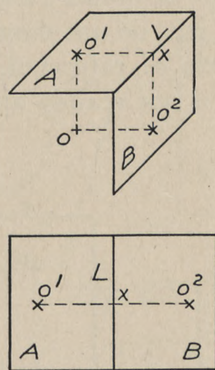

Fig. 21.

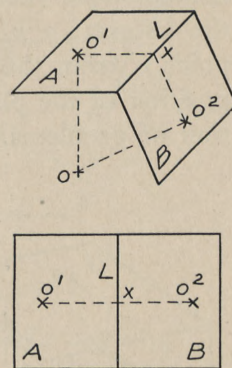

Fig. 22.

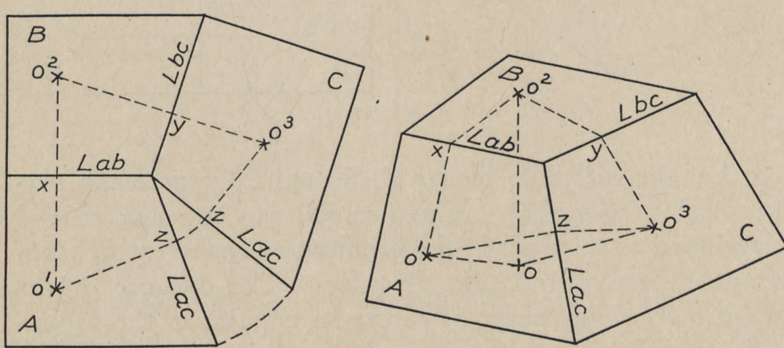

Fig. 23.

24. *Projection of a Point on Three Planes.* Let three reference planes, A, B, and C be taken. Call the intersection of A and B, Lab; of B and C, Lbc; of A and C, Lac. Let a point o be projected orthographically on the three planes at o^1, o^2, and o^3. Let the three planes be developed into a single plane surface by separating along the line Lac, and unfolding along the lines Lab and Lbc, as shown in Fig. 23. Then, by the preceding discussion, $o^1{\cdot}o^2$ is perpendicu-

lar to *Lab*, $o^2 \cdot o^3$ is perpendicular to *Lbc*, and the separated parts of $o^1 \cdot o^3$ are perpendicular, respectively, to the two positions of *Lac*. But if the three planes *A*, *B*, and *C* all make oblique (i.e., acute or obtuse) angles with each other, the distances $o^1 \cdot x$, $o^2 \cdot y$, etc., do not show the distances of the point *o* from the planes of reference. Consequently this position of the planes is not readily applicable to the solution of problems.

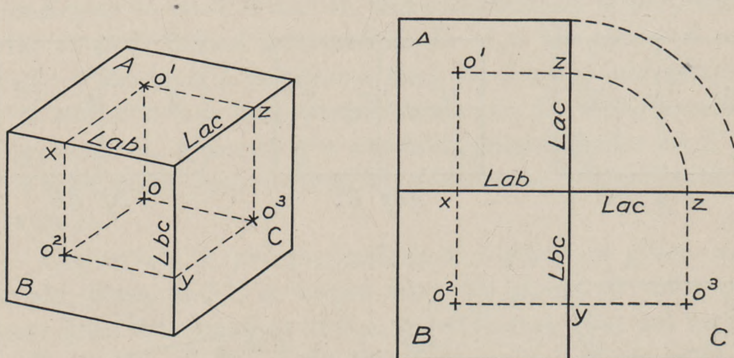

Fig. 24.

Let the reference planes *A*, *B*, and *C* be mutually perpendicular (Fig. 24). This position can be so taken and developed as to result in the familiar arrangement of plan, front elevation, and side elevation. The distance of the point *o* from any plane of reference appears in the projection on each of the other two planes. This is a useful and practical arrangement of the reference planes; nevertheless, there are limits to its application.

Let the three planes *A*, *B*, and *C* be taken as follows: *B* is perpendicular to *A*, and *C* is perpendicular to *B* but not to *A*. Let point *o* be orthographically projected at o^1, o^2, and o^3. Then let the planes be developed by separating along the oblique angle between *A* and *C*, and unfolding the two right angles between *A* and *B* and between *B* and *C*. Possible results are shown in Figs. 25 and 26. As in the

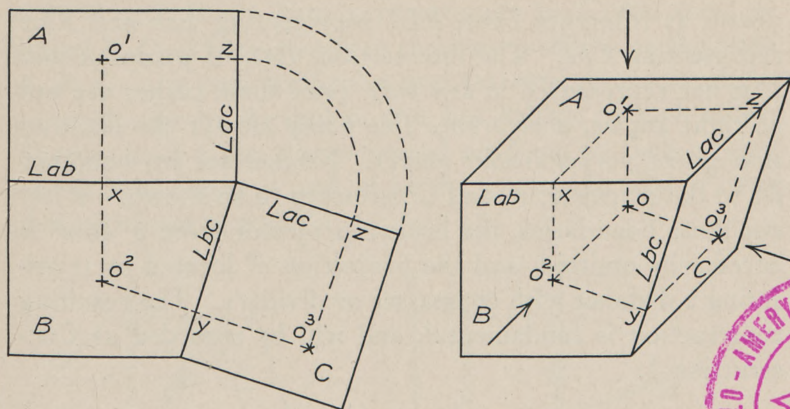

Fig. 25.

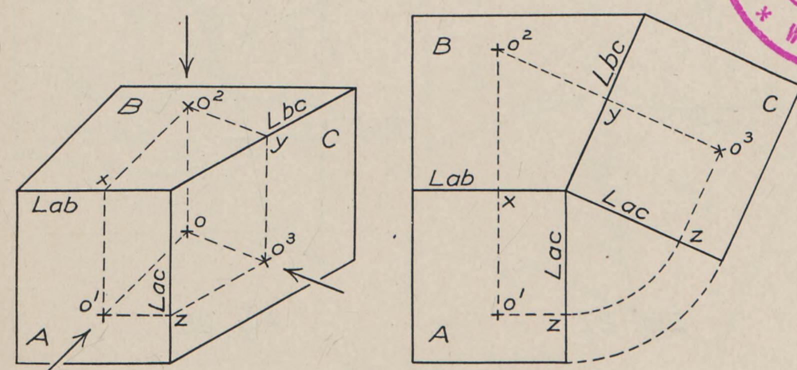

Fig. 26.

general case, Fig. 22, $o^1 \cdot o^2$ is perpendicular to *Lab*, $o^2 \cdot o^3$ is perpendicular to *Lbc*, and the projector $o^1 \cdot z \cdot z \cdot o^3$ has the parts $o^1 \cdot z$ and $z \cdot o^3$ each perpendicular to a position of *Lac*. However, in this case, since the planes *A* and *C* are both perpendicular to *B*, their line of intersection, *Lac*, is perpendicular to *B*. Whence the distance $o^1 \cdot x = o^3 \cdot y$, and each is equal to $o \cdot o^2$, or the distance of point *o*, in space, from the plane *B*.

The method of applying this principle to an actual drawing is shown in Figs. 27 and 28. The lines of intersection between the planes *A* and *B* and between *B* and *C* are

drawn as reference lines, *RL1* representing *Lab* and *RL2* representing *Lbc*. The intersection, *Lac*, of planes *A* and *C* is not represented in any way, since these planes are not at right angles, *a reference line being always the intersection of two perpendicular planes.* Neither are any boundaries to the planes *A, B,* and *C* necessary to be shown. Without these boundaries, the broken projector from o^1 to o^3 is necessarily omitted, and the projection o^3 located by transfering a distance with compasses or dividers. The resulting construction is fundamental, and will be recorded as Construction 1.

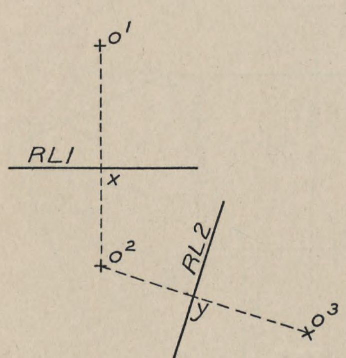

Fig. 27.

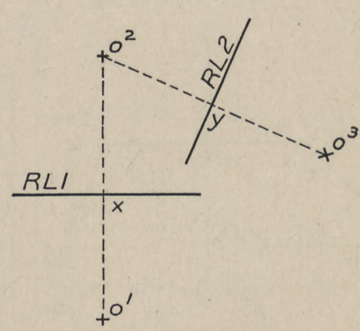

Fig. 28.

CONSTRUCTION 1. *Given two projections of a point, to construct a third projection of the point.*

See Figs. 27 and 28. Let o^1 and o^2 be the given projections, and *RL1* and *RL2* the given reference lines. To locate the projection o^3, project from o^2 perpendicular to *RL2*. On this projector, lay off the distance $y{\cdot}o^3$ equal to the distance $x{\cdot}o^1$.

25. *Projection of a Point on Four Planes.* The only case to be considered is the extension of the previous projection on three planes. As before, let plane *B* be perpendicular to *A,* and let *C* be perpendicular to *B* but not to *A*. Then let plane *D* be perpendicular to *C* but not to *B*. Then,

since B and D are both perpendicular to C, the *triad* of planes B, C, D is relatively in the same position as the *triad* A, B, C. In each triad, the two extreme planes are both perpendicular to the middle one, but oblique to each other. Therefore, if point o is projected on D at o^4, then o^4 may be obtained from o^3 and o^2 in the same way that o^3 was obtained from o^2 and o^1.

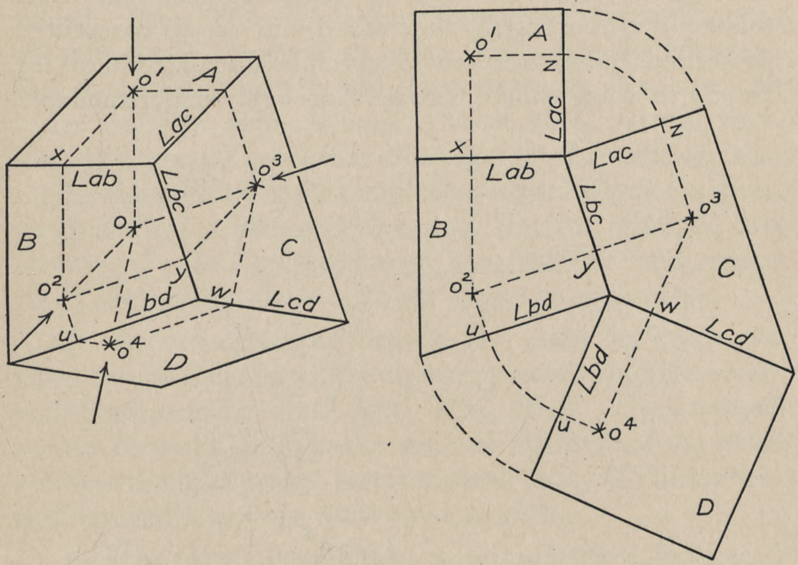

Fig. 29.

It is not an easy matter to visualize the four planes A, B, C, and D at one time, in all the infinite variety of positions that they may assume. A fairly easy position is shown in Fig. 29, together with the corresponding development. As with three planes, the separation is made along the oblique angles, and the unfolding along the right angles. *The actual method of projecting is shown in Fig 30, where o^4 is obtained by projecting from o^3 perpendicular to RL3, and making the distance $w{\cdot}o^4 = y{\cdot}o^2$.*

It should now be evident that this method is capable of indefinite extension. For example, plane E can be taken

perpendicular to D but not to C, and the *triad* C, D, E still has the fundamental position that the extreme planes are both perpendicular to the middle plane but not to each other. And so on. It is also evident that with each additional plane of projection, the difficulty of visualizing the entire series at once increases enormously. *Fortunately, such visualization is not necessary.* Each *triad* of consecutive planes is in the same fundamental position. Once this *triad* is thoroughly understood, any set of consecutive projections, no matter how long, can be visualized in threes, and by so doing should become thoroughly understandable.

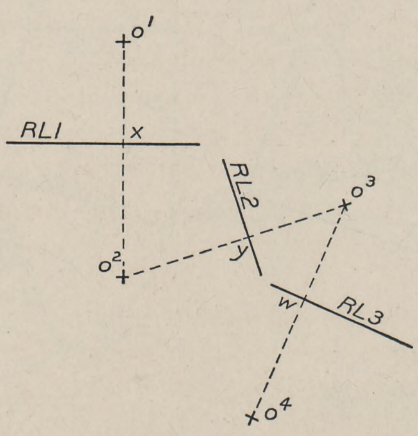

Fig. 30.

26. *Projection of the Straight Line and the Plane.* The method of successive projection is based on the foregoing construction. For, if one point can be successively projected in that manner, so can any fixed group of points which determine lines, surfaces, or solids, each point in the group being projected from view to view in the same way.

The first extension of the construction will be to the simplest possible groups, namely, two and three points, which may be considered as determining, respectively, a straight line and a plane.

27. *Notation for Successive Projections.* In solving problems by the method of successive projections, we shall call these projections simply the first, second, third, and fourth views. The two views, such as plan and elevation, which give the data, will be called the first and the second views. But which one is to be considered as the first view, will depend upon the direction from which the third view is taken. Thus, if the third view is projected from the plan, the series will be elevation, plan, third view, fourth view. But if the third view is projected from the elevation, the series becomes plan, elevation, third view, fourth view. This will greatly generalize the solution of problems; also render the solution easily adaptable to cases where the given views are not a plan and elevation, but, for example, two elevations. It is only necessary that the first and second views shall be on planes at right angles to each other.

In the accompanying figures, the order in which the views are considered will be shown by numbering the reference lines in order, thus, *RL1, RL2, RL3.* The first and second views lie on opposite sides of *RL1*; since the third and fourth views cannot be constructed until *RL1* is known, this line will always be considered as given.

Further, when no ambiguity will result, the indices used in Chapter 1, such as h, v, p, 4, etc., will be omitted. This can be done, since a fixed placing of the views, namely, plan above elevation, has been adopted.

28. *The Straight Line.* The mathematical straight line is of indefinite length. As the edge of a solid object, however, a line always terminates. We shall therefore take as the typical straight line the line ab, whose ends are the points a and b, with the understanding that the line may be produced in either direction if necessary.

29. *Projections of a Straight Line.* The simplest projection of a line is a point. This occurs when the line is perpendicular to the reference plane, and is called the end

view of the line. A second projection of the line on any reference plane perpendicular to the first shows the line in its actual or true length (*TL*), perpendicular to the reference line. See Fig. 31.

If one view of a line shows the true length, such as the plan in Fig. 32, and the other view is not a point, this view must be a line parallel to the reference line. For a line, in order to project in true length, must be parallel to the reference plane; that is, every point of the line is equally distant from the reference plane. A line in this position may be projected in end view, as a point, by a single additional view, as shown in Fig. 32, by taking a reference plane perpendicular to the true length view of the line.

The general position of a straight line is shown by several examples in Fig. 33. The views may make any angles with the reference line and with each other. In all of these examples the line is not parallel to either plane of reference, and neither view shows the true length of the line in space.

A line in the general position can always be projected in true length by a single additional view, and as a point by two additional views. These are the two fundamental constructions for the straight line, and will be recorded as Constructions 2 and 3.

CONSTRUCTION 2. *To find a true length projection of a straight line.*

See Fig. 34. Let *ab* be the given line. Project on any plane parallel to the given line. That is, assume a reference line parallel to either given view of the line, find the third projection of points *a* and *b* (Cons. 1), and connect the points. The resulting view is the true length of the line.

CONSTRUCTION 3. *To find the end view of a straight line.*

See Figs. 32 and 34. Find, if not already given, a true length view of the line (Cons. 2). Project on any plane per-

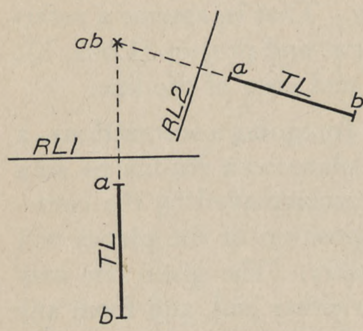

Fig. 31.

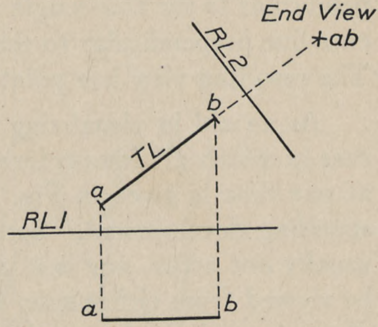

Fig. 32.

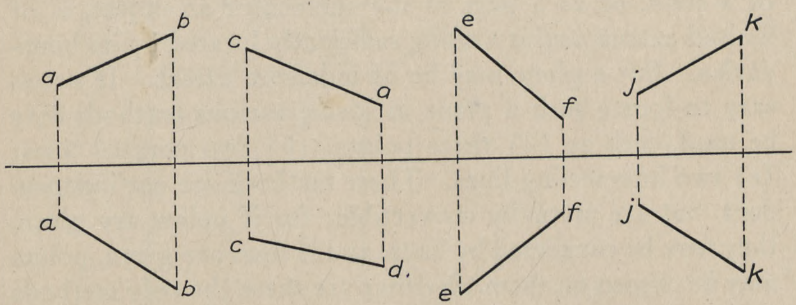

Fig. 33.

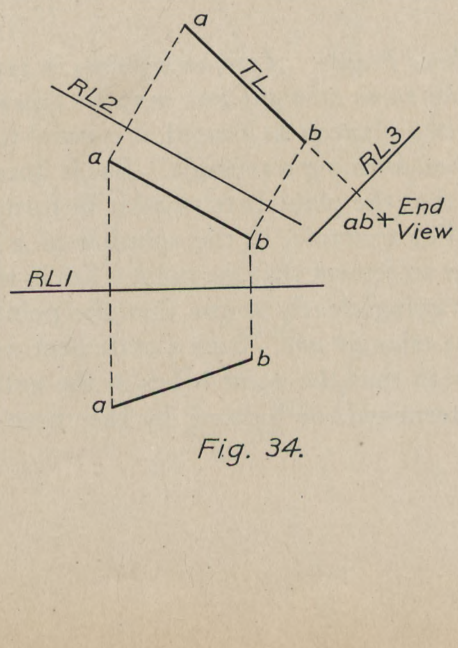

Fig. 34.

pendicular to the true length view. That is, assume a reference line perpendicular to this view, and project (Cons. 1). The resulting view is a point, or end view of the line.

As an aid in visualizing the foregoing constructions, a case in which all four reference planes can readily be seen at one time is given in Fig. 35, accompanied by the corresponding development. Such a position of the planes will usually not occur, nor is it necessary. The given line may be viewed from either side, from either end, and from any distance, as desired.

30. *Planes.* A plane which appears as one of the faces of a solid, or as a part of the surface of an object, is of limited extent, and is usually sufficiently located by its boundaries. But a plane may be of indefinite extent. If necessary to locate such a plane in space, various methods may be used, such as (*a*) three points; (*b*) two parallel lines; (*c*) two intersecting lines. These methods are not independent, but are mutually convertible; for if points are given, they may be connected by lines, and if lines are given, points may be chosen on them. Neither are these the only methods which may be used, but any other, such as a line and a point not on the line, may be readily converted into any one of them.

31. *Three Points. Any three points in space which do not lie on the same straight line may be taken to locate a plane.* When a plane is so located, it is usual to connect the points by lines, forming a triangle. But it does not follow, necessarily, that the plane is triangular in form and limited in extent. For example, in the solution of a problem we may have the statement that the point *o* lies in the plane *abc*, while the drawing clearly shows that the point *o* does not lie within the triangle *abc*. Such a statement must be interpreted to mean that the point *o* lies in the unlimited plane which is determined, or located, by the three points *a, b,* and *c*.

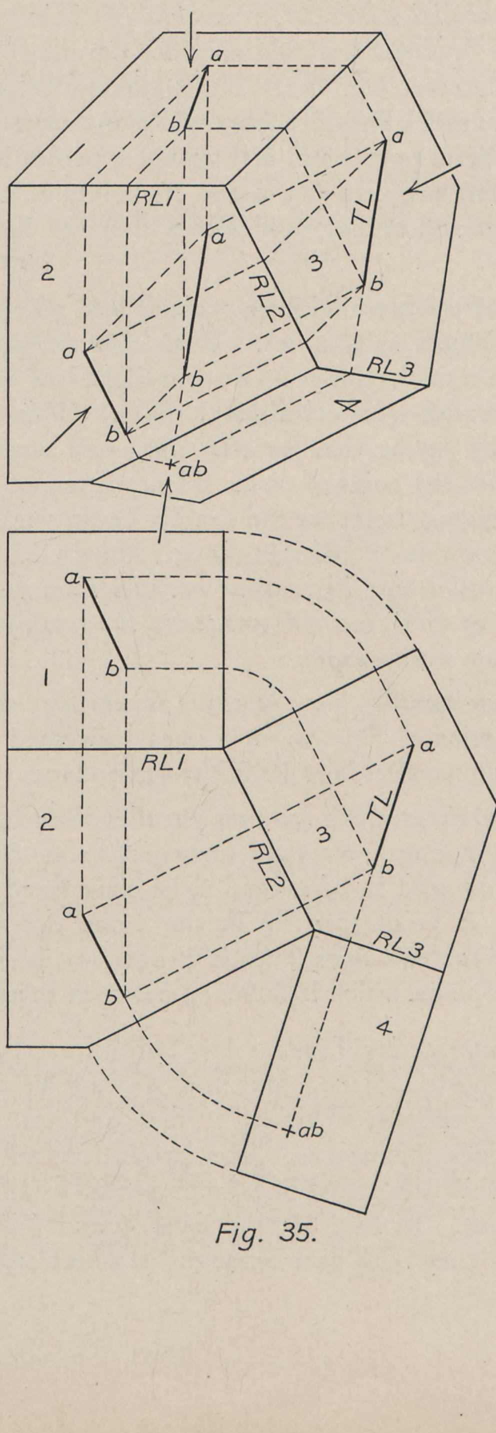

Fig. 35.

32. *Parallel Lines.* On account of their unique properties, the draftsman should consider parallel lines as distinct from intersecting lines. The mathematical generalization, that parallel lines are lines which intersect at infinity, serves no useful purpose in orthographic projection. Unless the entire line projects as a point, its points at infinity cannot be projected orthographically within the limits of the drawing.

If one of two parallel lines projects as a point, the other will also project as a point. With this exception, every orthographic projection of two parallel lines will consist of two parallel (or coincident) lines. This is usually expressed by saying that *parallel lines will always appear parallel from any point of view.* Hence if one of two parallel lines is given, the other can readily be drawn in any projection as soon as one of its points is known. *Lines which are not parallel may be seen as parallel lines from particular points of view, but will not stand the test of appearing parallel from every angle.*

Any two parallel lines in space locate a plane. There is no exception. This is sometimes expressed by saying that any two parallel lines lie in the same plane.

A plane determined by two parallel lines is obviously not limited in length, since the lines may be produced indefinitely. Nor need it, of necessity, be considered as limited in width. A point may lie in the plane determined, or located, by the two lines, even if it does not lie in the portion of the plane which is included between them.

33. *Intersecting Lines.* Any two intersecting lines in space will locate a plane. But in a drawing we are dealing with the projections, or views, of lines, and not, in general, with the lines themselves. The projections of lines may intersect, while the actual lines, represented by those projections, do not. In Fig. 36, the lines A and B intersect in space, since they both pass through the point e; these lines,

therefore, locate a plane. But the lines C and D have no point in common; therefore they do not intersect, and since they are obviously not parallel, they cannot locate a plane.

Hence, when a plane is to be located by two intersecting lines, care must be taken that the lines actually intersect. But when a plane is so located, it is readily apparent that these lines may be produced indefinitely, and consequently that the plane is unlimited in extent.

34. *Lines in a Plane.* As soon as two lines in a plane are known, any number of lines lying in the plane may be drawn. But, in general, *only one view of an additional line can be assumed*; the other view must be found by projection.

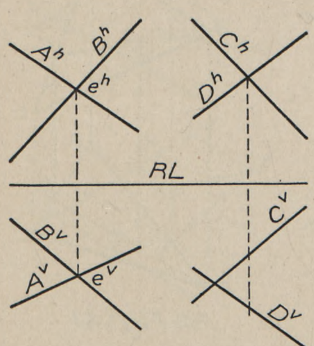

Fig. 36.

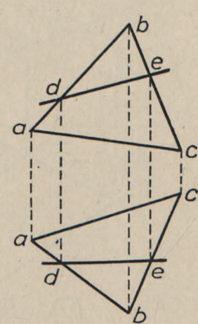

Fig. 37.

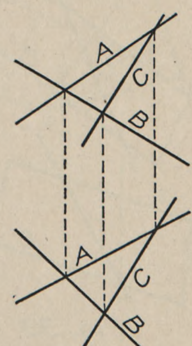

Fig. 38.

LEMMA 1. *Given one projection of a line lying in a plane, to find the other projection of the line.*

Case 1. The line intersects two known lines in the plane. Project the intersections to the other view, and connect.

Case 2. The line intersects a known line of the plane, and is parallel to a second known line. Project the point of intersection, and draw parallel to the second line.

The first case is shown in Figs. 37, 38, and 39. In Fig. 37, the plane is given as a triangle, and the line *de* is located by its intersections with the sides *ab* and *bc* of the triangle.

In Figs. 38 and 39, the plane is given by two lines *A* and *B*, and the line *C* is located by its intersections with *A* and *B*.

The second case is shown in Fig. 40. The line *C* is located as intersecting *A* and parallel to *B*. In all four of these figures, it is evident that either view may represent the given projection, and the other the required projection.

If either of these methods fail, due to points falling outside the limits of the drawing, or to the position of the data, assume one or two points, as necessary, on the given line, and use the following corollary.

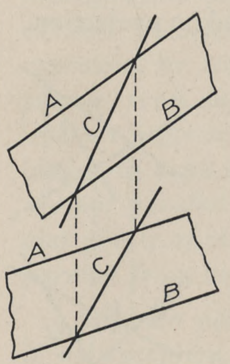

Fig. 39.

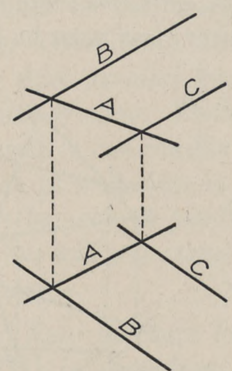

Fig. 40.

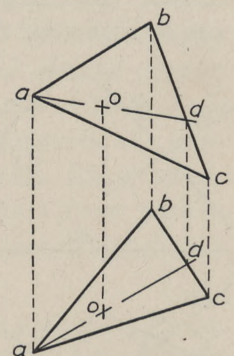

Fig. 41.

COROLLARY. *Given one projection of a point lying in a plane, to find the other projection of the point.*

See Fig. 41. Let *o* (either projection), lying in the plane *abc*, be the given point. Through the given projection of *o*, draw a line whose second projection can be readily found; the line here shown is *ad*. The required projection of *o* lies on the second projection of *ad*.

This method can be extended to project any number of points in a plane, as shown in Fig. 42. The auxiliary lines drawn intersect *A* and are parallel to *B* (compare Fig. 40). Here the points *d* and *e* could not have been found by producing the line *de* to its intersections with *A* and *B* without the aid of an additional view.

Note. Since a straight line is often defined as the shortest distance between two points, the beginner is thereby sometimes led to consider that the only way to determine a straight line, or its projections, is to locate two points. Hence he is always looking for intersections. But parallel lines do not intersect. *Much time and much accuracy may readily be lost by failing to note that, if two lines are parallel, and one of the lines is known, only one point is needed to locate the second line.*

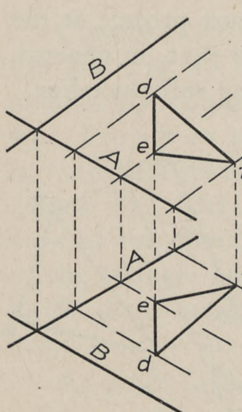

Fig. 42.

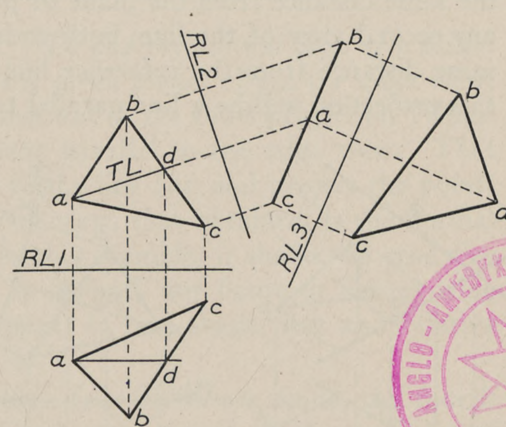

Fig. 43.

35. *Edge View of a Plane.* The simplest projection of a plane is a straight line. This view results when the given plane is perpendicular to the plane of projection, and is known as the edge view of the plane.

The edge view of a plane occurs when some line in the plane projects as a point. For, if a line projects as a point, it is perpendicular to the plane of projection; and if a line is perpendicular to a plane, any plane which contains the line is perpendicular to the first plane.

Since any line can be projected as a point by two additional views (Cons. 3), it follows that any plane can be projected as a line by two additional views. *But by choosing particular lines in the plane, any plane can be projected*

as a line by one additional view. The lines which must be chosen are those which can be reduced to a point by one additional view, namely, lines which are projected in true length in one of the given views.

LEMMA 2. *In a given plane, to draw a line, one of whose projections shall be a true length view of the line.*

If a line projects in true length, it is because the line is parallel to the plane of projection; that is, both ends are at the same distance from the plane of projection. Hence, in any second view of the line, both ends must project at the same distance from the reference line; so that, in general, this projection will be a line parallel to the reference line.

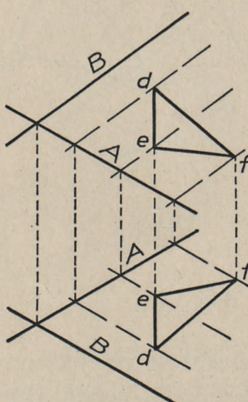

Fig. 42.

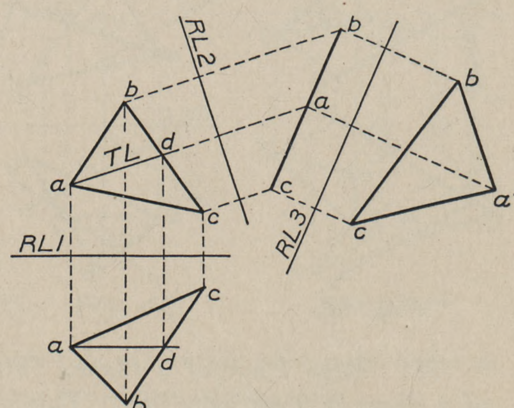

Fig. 43.

Let *abc,* Fig. 43, be the given plane. In either view draw a projection, as *ad,* parallel to the reference line. Consider this as one projection of a line lying in the plane, and find the other projection (Lemma 1). The projection thus found is in true length, and is the required projection.

CONSTRUCTION 4. *To find the edge view of a plane.* Let *abc,* Fig. 43, be the given plane. In the plane, draw a line, as *ad*, one of whose projections is in true length (Lemma 2). Project on a plane perpendicular to *ad.* That

is, assume *RL2* perpendicular to the true length view of *ad,* and project the various points (Cons. 1). Since *ad* projects as a point, all the points of the plane must fall in a straight line, which is the edge view required.

36. *Normal View of a Plane.* A normal view of a plane is a view taken perpendicular to the plane. In other words, it is a projection of the plane on a reference plane parallel to the given plane. As soon as an edge view of a plane is found, a normal view can be immediately obtained.

CONSTRUCTION 5. *To find the normal view of a plane.* Let *abc,* Fig. 43, be the given plane. Find the edge view, namely, the straight line *bac* in the third projection (Cons. 4). Project on a plane parallel to the edge view. That is, take *RL3* parallel to the line *bac,* and project the points (Cons. 1). The fourth view, thus obtained, is the normal view of the plane. In this example, it shows the true size of the triangle *abc.* In general, any figure or figures lying in the plane would appear in true shape, size and relative position.

As in all cases where four views are made, the fourth reference plane is placed with respect to the second and third planes, and with no regard to the position of the first plane. Consequently, as with the four views of a straight line, situations in which all four reference planes can easily be seen at once are the exception, rather than the rule. But as an aid in visualizing Constructions 4 and 5, a position, different from that used in Fig. 43, is shown pictorially in Fig. 44a, and in development in Fig. 44b. The amount of space between the various views is excessive for an actual drawing, but was needed to keep the points clear in the pictorial view.

37. *Summary of the Point, Line, and Plane.*

A point will always project as a point, no matter how many views are taken.

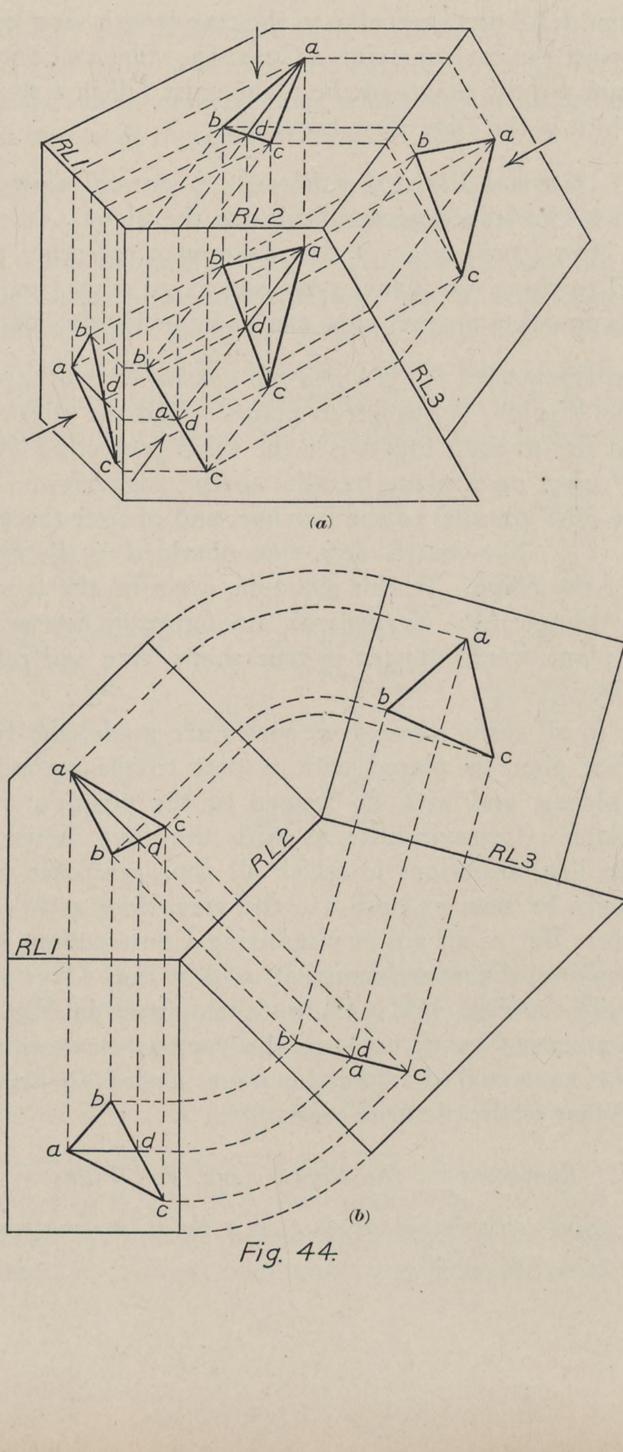

Fig. 44.

A straight line, given in a general oblique position in each of two views, can always be projected in true length in a third view, and as a point in a fourth view.

A plane, given in a general oblique position in each of two views, can always be projected as a straight line in a third view, and normally in a fourth view.

Chapter 3

Problems on the Point, Line, and Plane

38. In this chapter, the constructions of the preceding chapter will be applied to the solution of problems involving points, lines, planes, and some of the simpler solids, such as prisms and pyramids. Where solids are employed, attention should be paid to the visibility of their edges.

39. *Visibility of Solid Objects.* The visibility of an object as shown in its various views should be consistent throughout. Thus, consider three consecutive views, such as the second, third, and fourth; the visibility shown in the third view should be the same, whether obtained from the second view or the fourth view. Similarly, with the first, second, and third views, the visibility of the second view, if obtained from the third view, should be the same as that obtained from the first view.

Since we have decided to work only in the third quadrant (Art. 14), this result may be accomplished by using the following special rule for visibility. *In the third quadrant, either projection is a view of the other projection as seen by looking from the reference line.* Hence for every pair of views separated by a reference line, each view should be visualized as if looking at the other view from that particular reference line.

As previously stated, the views in engineering drawing are usually placed according to third quadrant methods, and hence are visualized in the above manner. Consequently this, or a similar rule for visibility, is often given as if it were universal. But this is not so; this special rule applies only to the third quadrant. For the general rules for determining visibility in any quadrant, see Art. 10.

40. *Method of Attack.* In the solution of a problem which is not a direct application of one of the fundamental constructions, the following method of attack may be used. First, obtain a solution by placing data in the simplest possible position; this will be called the basic solution. Then, in the general case, where the data are not so placed, find a method of reducing this case to the basic solution by suitably chosen additional views.

41. *Problems Involving the True Length of a Straight Line.*

PROBLEM 1. *At a given point in a line, to draw a plane perpendicular to the line.*

Let *ab* be the given line, and let the plane be required to contain point *a*. Let the line be given as in Fig. 45, in which neither given view is in true length. Find a third view showing the true length (Cons. 2). In the third view, the plane will appear edgewise, as the line *dac* perpendicular to *ab*.

To locate the plane as a triangle in the original views, select any two points *d* and *c* in its edge view. Project back to the second view, selecting any two points in line with those in the third view. Project to the first view, locating *c* and *d* by transferring distances from the third view (Cons. 1). The triangle *adc,* drawn in the original views, represents the plane.

PROBLEM 2. *At a given point in a line, to draw a line perpendicular to the given line.*

Let *ab* be the given line, and *ac* the required line. Lines *ab* and *ac* are to be perpendicular to each other at point *a.*

General Case. See Fig. 45. The required perpendicular will lie in the plane perpendicular to the given line at point *a.* Find a true length projection of *ab,* and draw the edge view of the perpendicular plane, as in Prob. 1.

Select any point *c* in the plane, and project this point to the original views. Line *ac* is the required line.

Special Case. One view of the line *ab* is given in true length. See Fig. 46. In this figure, we see that the elevation of *ab* is parallel to the reference line. Hence the line *ab* in space is parallel to the *H*-plane of projection, and its plan is in true length. A plane perpendicular to *ab* will therefore show in edge view in the plan. Therefore, in the plan, draw *ac* perpendicular to *ab*. Select any point *c* and project to the elevation, in which view *c* may be represented by any point on the projector. Thus, ac^1, ac^2, or ac^3, taken with *ac* as the plan, all represent lines perpendicular to *ab*.

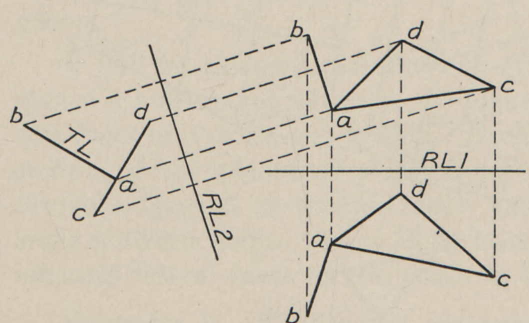

Fig. 45.

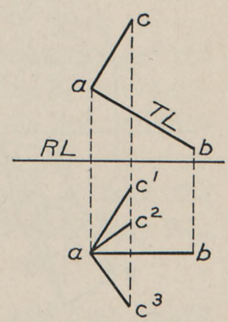

Fig. 46.

42. *Perpendicular Lines.* In Fig. 46 we see that the line *ac* projects at right angles to *ab* in the plan, but *ac* is not in true length in the plan unless its elevation coincides with the elevation of *ab*. This is not necessarily the case. That is, two lines which are at right angles in space will project at right angles in a view in which only *one* of them projects in true length. Hence the following rule for perpendicular lines.

If two lines in space are at right angles to each other, their projections will be at right angles if one of the lines projects in true length.

This rule is of wide application in the projection of solids whose edges are at right angles.

APPLICATION. *To find the projections of the right section of an oblique prism.*

By an oblique prism is meant one whose ends are not perpendicular to the axis, nor necessarily parallel to each other. A right section is a section perpendicular to the axis, or, what is the same thing, to the lateral edges.

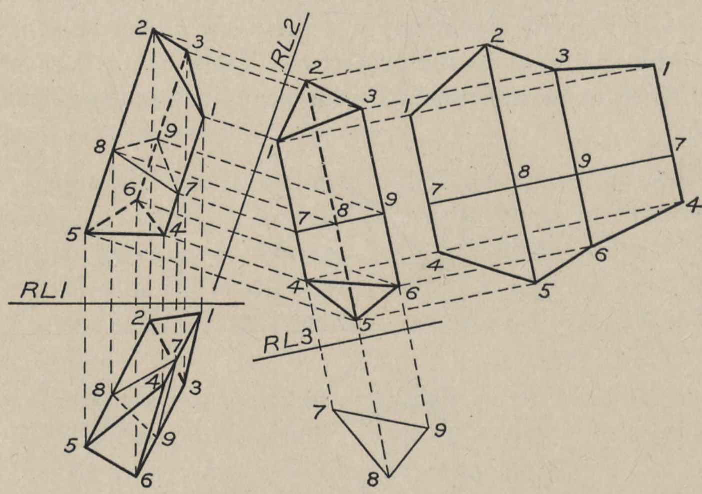

Fig. 47.

See Fig. 47. Since the lateral edges are all parallel, a third view which shows one edge in true length shows them all in true length. A plane at right angles to the lateral edges appears in this view as a straight line perpendicular to the edges (Prob. 1). This plane may be drawn anywhere between the two ends of the prism. After the plane is drawn, the points in which it cuts the lateral edges are projected back to the original views, as shown.

43. *Problems Involving the End View of a Line.*

PROBLEM 3. *To find the shortest distance between two parallel lines.*

If one of the lines is projected as a point (Cons. 3), the other line will also project as a point. The distance between these points is the true distance between the given lines.

APPLICATION. *To find the true right section of an oblique prism.*

See Fig. 47. Since all the lateral edges are parallel, a fourth view can be obtained in which all the edges project as points (Prob. 3). When this is done, the polygon formed by joining these points is the true cross-section of the prism.

PROBLEM 4. *To find the true angle between two planes.*

It will be assumed that the line of intersection of the planes is known; for example, the planes may be two adjacent faces of a prism or pyramid. Project the line of intersection of the planes as a point. The planes will then project edgewise, as straight lines, and show at once the angle between them. As an illustration, consider any two adjacent lateral faces of the prism in Fig. 47.

PROBLEM 5. *To find the projections and true length of the shortest distance from a point to a line.*

Let *ab* be the given line, and *c* the given point. Let *d* be the point in which the perpendicular from *c* intersects the line *ab*, so that it is required to find the projections and true length of the line *cd*.

Basic Solution. See Fig. 48. In the first view, let *ab* project as a point. Then in this view the projection *cd* can be drawn at once, from *c* to *ab*. In the second view, *ab* projects in true length. Hence in this view the projections of *ab* and *cd* are at right angles (Art. 42). Since, in this view, *ab* is perpendicular to the reference line, it follows that *cd* is parallel to the reference line. Hence the first view of *cd* is its true length.

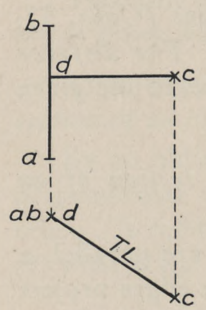

Fig. 48.

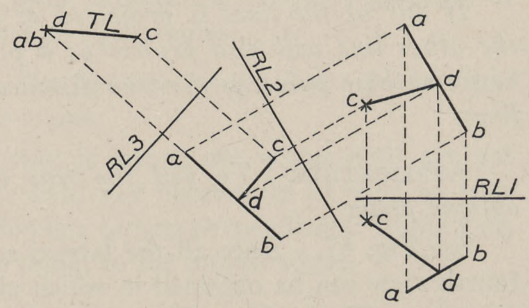

Fig. 49.

General Case. See Fig. 49. Find a third view, in which *ab* projects in true length, and a fourth view, in which *ab* projects as a point. In the fourth view, *d* coincides with *ab,* and the projection *cd* is the true length of the required perpendicular. In the third view, *cd* and *ab* are at right angles.

PROBLEM 6. *To find the projections and true length of the shortest distance between two lines not in the same plane.*

The shortest distance between two lines not in the same plane is the line which is perpendicular to each of them. Let *ab* be one of the given lines, and *cd* the other. The required line is then perpendicular to both *ab* and *cd.* Let this line be *ef,* from point *e* on *ab* to *f* on *cd.* It is required to obtain the projections and true length of the line *ef.*

Basic Solution. See Fig. 50. Let one view of *ab* be a point; then the projection of *e* coincides with *ab.* In this view, the shortest distance from point *e* to any point on *cd* is obviously the perpendicular, *ef,* from *e* to *cd.* This locates *f* on *cd*; project to the second view. In the second view, a perpendicular from *f* to *ab* must project at right angles to *ab,* since in this view *ab* is in true length (compare Fig. 48). The projections of *ef* have thus been obtained; and since the second view of *ef* is parallel to the reference line, its first view shows its true length.

General Case. See Fig. 51. Project one of the given lines as a point (Cons. 3); then apply the basic solution.

44. *Problems Involving the Edge View of a Plane.*

PROBLEM 7. *To find the intersection of a line and a plane.*

See Fig. 52. Find a third view, in which the plane projects edgewise (Cons. 4). Project the line into the third view; then the point in which the line intersects the plane appears at once.

PROBLEM 8. *To find the shortest distance from a point to a plane.*

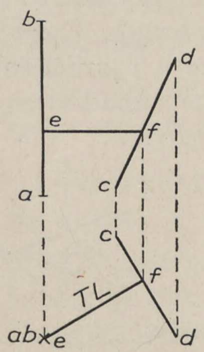

Fig. 50.

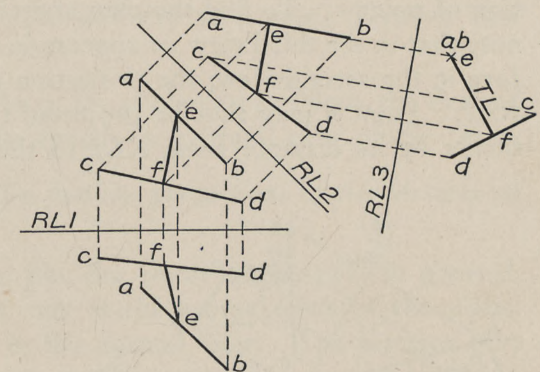

Fig. 51.

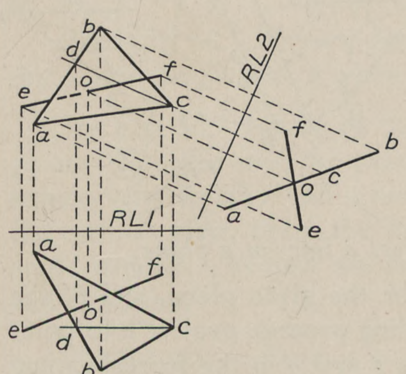

Fig. 52.

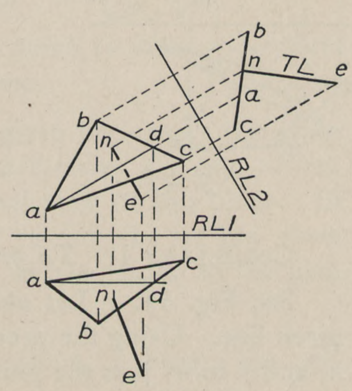

Fig. 53.

See Fig. 53. Let *abc* be the given plane, and *e* the given point. Find a third view, in which the plane appears edgewise (Cons. 4). Project point *e* to this view; then the perpendicular *en,* from *e* to the plane, is the required shortest distance.

If the plan and elevation of this perpendicular are required, see the next problem.

PROBLEM 9. *To project a point on a plane.*

See Fig. 53. Proceed as in Prob. 8, and obtain in the third view the point *n* in which the perpendicular from *e* intersects the given plane. Point *n* is the required projection of point *e.* To find the plan and elevation of point *n,* note that in the third view *en* appears in true length. Therefore in the second view, the projection of *en* is parallel to *RL2.* Point *n* may thus be projected to the second view, thence by its distance from *RL2* to the first view.

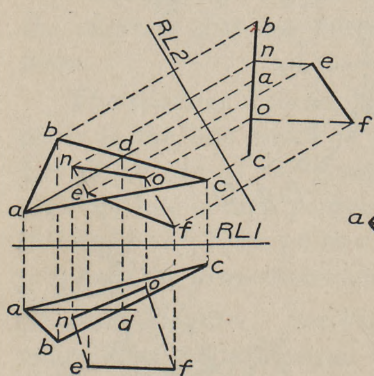

Fig. 54.

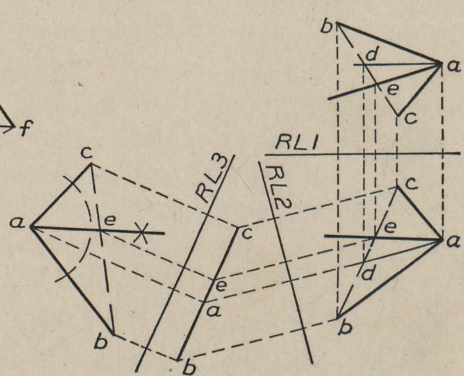

Fig. 55.

COROLLARY 1. *To project a line on a plane.*

See Fig. 54. Let *abc* be the given plane, and *ef* the given line. Using the preceding process, project *e* to point *n,* and *f* to *o*; join the points *n* and *o* in the corresponding views. Line *no* is the required projection of *ef.*

COROLLARY 2. *Through a given line, to pass a plane perpendicular to a given plane.*

See Fig. 54. The plane *efon* is obviously the required plane, since it contains a line (*en* or *fo*) which is perpendicular to the given plane.

45. *Problems Involving the Normal View of a Plane.*

PROBLEM 10. *To find the true size of a plane figure.*

See Fig. 43. This problem is a direct application of Construction 5, the result in the fourth, or normal view, being, as there stated, the true size of the triangle *abc*.

PROBLEM 11. *To find the true angle between two intersecting lines.*

See Fig. 55. As in the preceding problem, the desired result appears in the fourth, or normal, view of the plane.

COROLLARY. *To find the projections of the bisector of the angle.*

The half angle, like the whole angle, will, in general, appear in true size only in the normal view of the plane. Draw the bisector in the normal view. The bisector may then be found in plan and elevation by carrying back any point, such as *e*.

PROBLEM 12. *From a given point, to draw a line making a given angle with a given line.*

See Fig. 56. Let *ab* be the given line, and *c* the given point. *The line and the point determine a plane.* Find, successively, an edge and a normal view of this plane. In the normal view, draw the line *cn* (or *co*) at the given angle with *ab*. Project the line *cn* (*co*) back to the plan and elevation.

COROLLARY. *From a given point, to draw a line perpendicular to a given line.*

In Fig. 56, in the normal view the line *cn* could as well have been drawn perpendicular to *ab* as at any other angle. This gives another method of solving Problem 5.

PROBLEM 13. *To construct a figure of a given shape and size in a given plane.*

See Fig. 57. In this figure the given plane, indefinite in extent, is located by the lines *cd* and *ce,* and *cd* is to be the side of a square lying in the plane.

Find a third view, showing the edge view of the given plane. Find a fourth view, showing a normal view of the given plane. In the fourth view, complete the plane figure as required. Project back to the original views.

APPLICATION. *To construct a right solid (prism, pyramid, cylinder, or cone) whose base lies in a given plane.*

See Fig. 57. The base is constructed as in the preceding problem. The altitude shows in the third view, perpendicular to the base.

PROBLEM 14. *To find the angle between a line and a plane.*

The angle between a line and a plane is the angle which a line makes with its own projection on the plane. A direct solution may therefore be had by combining Problem 9, Cor. 1, and Problem 11.

See Fig. 58. Let *abc* be the given plane, and *ef* the given line.

First. To project the line *ef* on the plane *abc.* Find a third view, showing the edge view of *abc.* Find also the view of the line *ef.* From *e* and *f,* draw *ej* and *fk* perpendicular to *abc.* Then *jk* is the projection of *ef* on the plane. (Compare Fig. 54). The plan and elevation of *jk* need not be found.

Second. To find the true angle between *ef* and *jk.* Since *ej* and *fk* project in true length in the third view, an

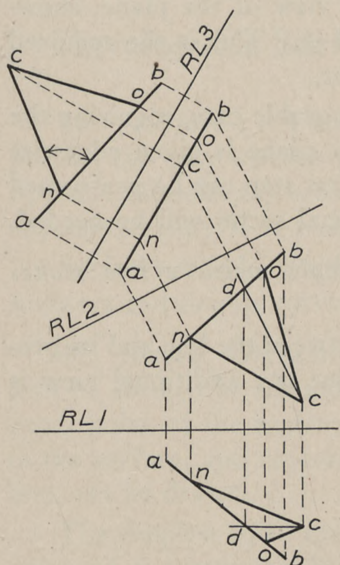

Fig. 56.

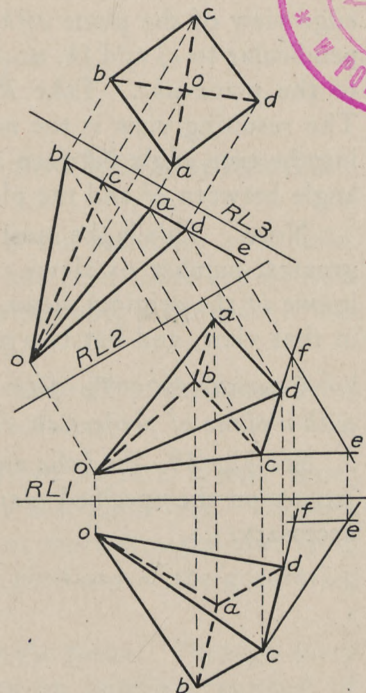

Fig. 57.

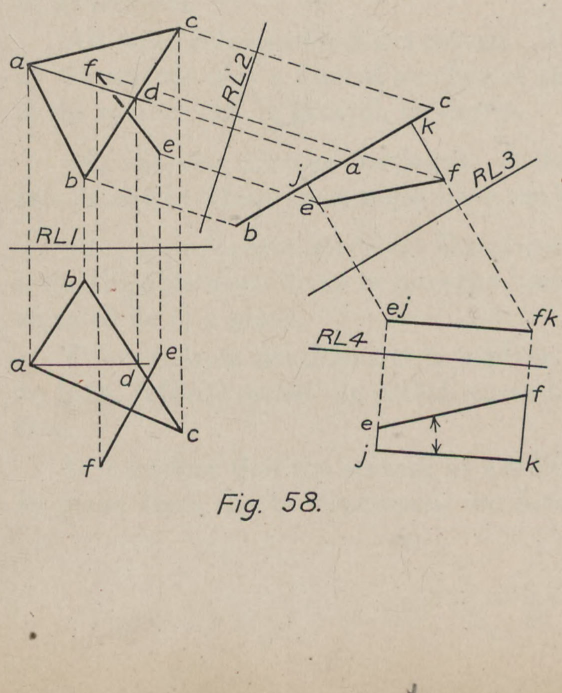

Fig. 58.

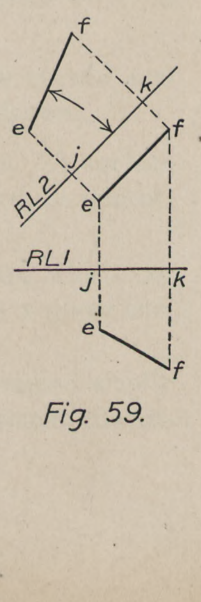

Fig. 59.

edge view of the plane *ejfk* will result by taking *RL3* perpendicular to *ej* and *fk*, or, what is the same thing, parallel to the plane *abc*. Take *RL4* parallel to the plane *efjk*. The resulting view is the normal view of the plane, showing the true angle between *ef* and *jk*. This is the required angle between *ef* and the plane *abc*.

Note. This is the most unfavorable case, requiring the greatest number of views. If the plane is given edgewise in one of the original views, the first step can be performed in that view, and but two additional views will be needed.

COROLLARY. *To find the angle which a line makes with a plane of projection.*

See Fig. 59. Find the angle between the line and its projection on the specified plane. But one additional view is necessary.

Chapter 4

Developable Surfaces and Developments

46. *Surfaces.* A surface is the boundary of a solid. So far, the only surface which has been conceived to exist by itself, without any accompanying solid, is the plane. But any surface may be so considered.

Surfaces may be divided into two general classes, ruled surfaces and double curved surfaces.

47. *Ruled Surfaces.* A ruled surface is one which may be formed by the motion of a straight line. Through every point of the surface, one or more straight lines lying in the surface, and representing positions of the generating line, can be drawn.

The simplest ruled surface is the plane. Through every point of a plane can be drawn an indefinite number of straight lines, representing an indefinite number of methods of generation.

All other ruled surfaces are curved. Through any point of the surface only a limited number of straight lines lying in the surface can, in general, be drawn.

48. *Double Curved Surfaces.* A double curved surface is one in which no straight lines can be drawn.

49. *Developable Surfaces; Developments.* A surface is developable when it can be unrolled, unfolded, or unbent, so as to lie in a plane.

Conversely, a development is a plane figure which can be bent, folded, curled, or rolled, so as to fit a given surface.

It is evident that the surface of a solid bounded wholly by plane faces can be developed, the development consist-

ing of the true sizes of the faces arranged in consecutive order. For example, the development of a cube consists of six squares, so arranged as to fold up and cover the cube.

A curved surface is developable only when it is a ruled surface in which each two consecutive positions of the generating line lie in the same plane; that is, are either parallel or intersecting. This condition is mathematically fulfilled by the consecutive tangents to any curve; and when the curve is such that a plane does not result, a developable curved surface is formed. Such a surface is known as a convolute. Convolutes are, however, rarely met with in practice.

The only other developable curved surfaces, and the only ones commonly used, are the cylinder, in which all positions of the generating line are parallel, and the cone, in which all positions intersect in a common point.

50. *The True Length of a Line.* A development, being a plane figure, is usually built up from the true lengths of lines. The true length of a straight line, if not shown directly in either plan or elevation, can always be found by one additional view (Cons. 2). But the lines whose true lengths are wanted may be numerous, and inclined at varying angles. Such, for example, might be the edges of an irregular pyramid, whose axis is not at right angles to the base. While a separate projection may be made for each line concerned, a shorter construction will be useful.

Before making any developments, therefore, we will consider the following auxiliary problem.

LEMMA 3. *To find the true length of a straight line as the hypothenuse of a right triangle.*

See Fig. 60. Let the line be *oa,* projected in plan as $o^h a^h$ and in elevation as $o^v a^v$, neither view showing the true length of the line.

Draw *RL1* through the upper end of the elevation, and take *RL2* coincident with the plan. Find the true length

$o^h a^3$ by Construction 2. We then observe that $o^h a^3$ is the hypothenuse of a right triangle whose base, $o^h a^h$, is the plan of the line, and whose altitude, $a^h a^3$, equals the vertical distance $a^v x$. Expressed in words, this becomes:

The true length of a line is the hypothenuse of a right triangle, whose base is the length of its plan, and whose altitude is the difference in elevation between the two ends of the line.

A similar result may be obtained by taking the elevation, or indeed any view of the line, as the base of a right triangle, with a corresponding altitude taken from another view. But in such cases the formula cannot be so readily expressed in words.

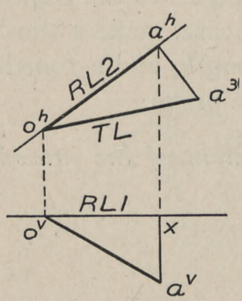

Fig. 60.

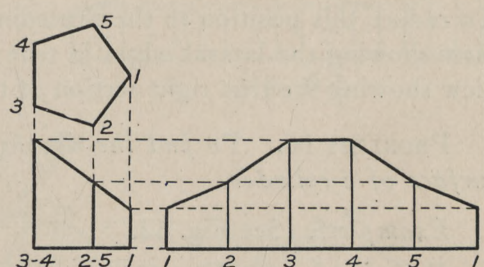

Fig. 61.

51. *True Length Diagram.* The actual length of the line being all that is desired, the two perpendicular sides of the triangle may evidently be laid off on any right angle. So that, in making a development, one conveniently placed right angle may be utilized in finding the true lengths of all the lines necessary. Such a figure, if drawn, is known as a true length diagram. A true length diagram is particularly useful if, as is often the case, the triangles can be so arranged as to have a common altitude.

52. *Problems in Development.* In each of the following problems, two examples will be given. First, a

simple case, in which all the true lengths needed can be obtained from the plan or elevation, followed by a general case, requiring a maximum amount of construction.

PROBLEM 15. *To find the development of the lateral faces of a prism.*

Example 1. See Fig. 61. The base of the prism is at right angles to the edges, and develops in a straight line *1·2···5·1*. The distances on this line are taken from the true size of the base, which appears in the plan. The lengths of the edges are in the elevation.

Example 2. See Fig. 47 and Prob. 3. Neither plan nor elevation gives the true lengths of the lateral edges, and neither end of the prism is at right angles to the edges. To reduce this position to the preceding case, make a third view showing the lateral edges in true length, and a fourth view showing the true right section of the prism.

PROBLEM 16. *To find the development of the curved surface of a cylinder.*

Example 1. See Fig. 62.

Example 2. See Fig. 63.

Draw a number of straight lines, or elements, on the surface. Treat these elements as the edges of a prism, and develop by the methods of the preceding problem, noting however that the points obtained are to be connected by curves and not by straight lines.

Since, in rectifying the right section, the approximation used is that the chord equals the arc, in order to obtain an accurate result the number of elements chosen must be great enough to justify this approximation. Also, whenever easily possible, the elements should be equally spaced, since the resulting symmetry makes the construction easier.

PROBLEM 17. *To find the development of the lateral faces of a pyramid.*

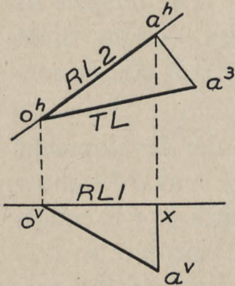

Fig. 60.

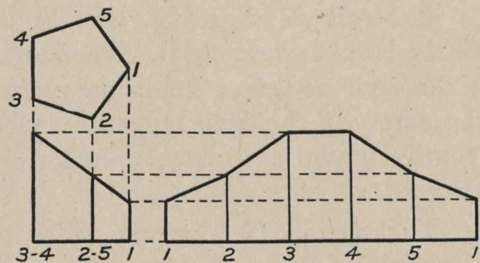

Fig. 61.

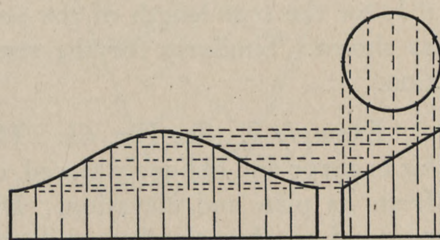

Fig. 62.

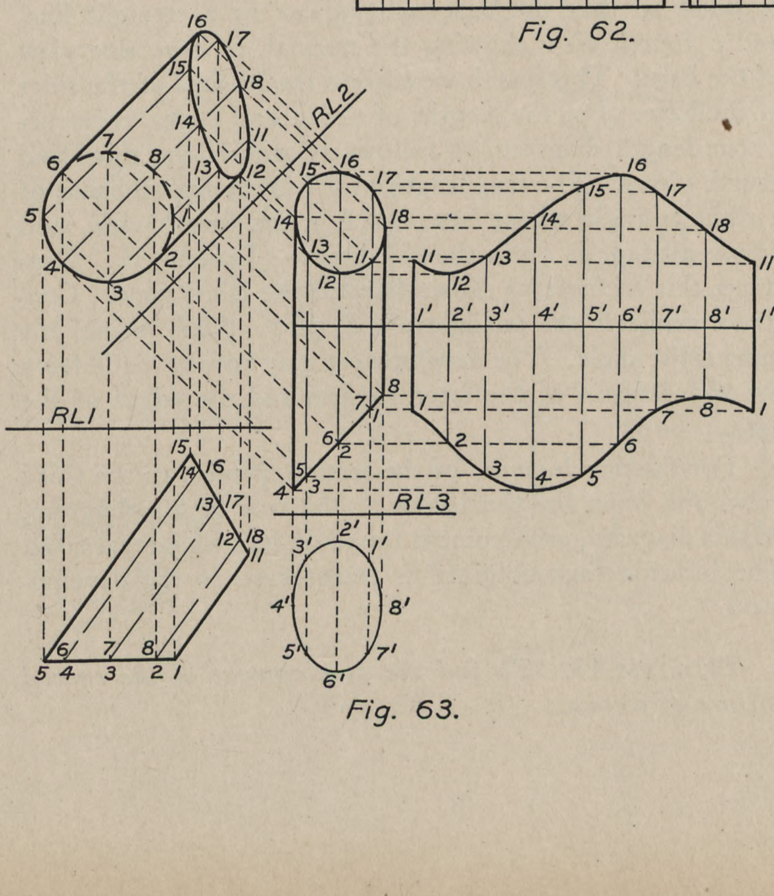

Fig. 63.

Example 1. See Fig. 64. A regular pyramid, with all the lateral edges equal. The length of each lateral edge is the same as *0·1* or *0·4* in the elevation. The lengths of the sides of the base appear in the plan. From these lengths, the series of equal triangles *0·1·2, 0·2·3*, etc., is constructed.

The section *11·12*···· may be located in the development by the true lengths of *0·11, 0·12,* etc. *0·11* and *0·14* show at once in the elevation. Since the true length of *0·2* equals *0·1*, for the true length of the portion *0·12* project to *0·1*, as shown. Similarly for the remaining points of the section.

Example 2. See Fig. 65. This pyramid, being irregular in shape and obliquely placed, cannot easily be developed from its plan and elevation. Draw a third view of the pyramid so that the base will appear as a straight line, and a fourth view showing the normal, or true size view of the base. The fourth view gives the lengths of the sides of the base. For the lengths of the lateral edges, construct a true length diagram, as follows: Consider the third and fourth views. The true length of *0·1* is the hypothenuse of a right triangle whose base is *0·1* in the fourth view, and whose altitude is *0x* in the third view (Lemma 3). Construct this triangle as shown beyond the third view. Construct similarly the triangles *0·x·2*, *0·x·3*, *0·x·4*, using the same right angle. The development may now be made from the true lengths of the lateral edges and the edges of the base.

For the development of the section, project to the third view, and from the third view to the true length diagram. In this diagram, each point is found on its own lateral edge. The location thus obtained is transferred to the development.

PROBLEM 18. *To find the development of the curved surface of a cone.*

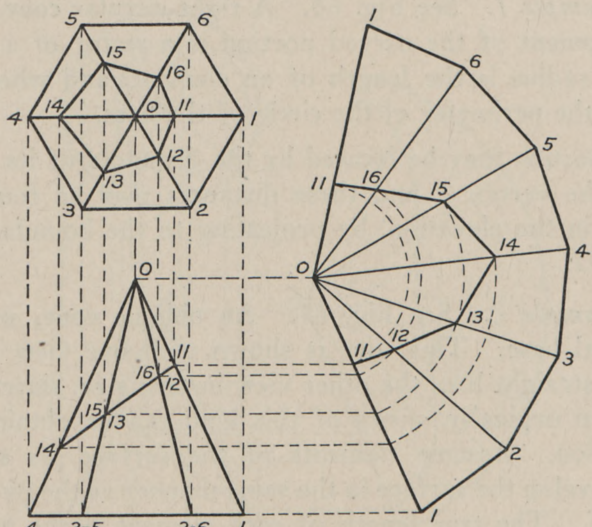
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Fig. 64.

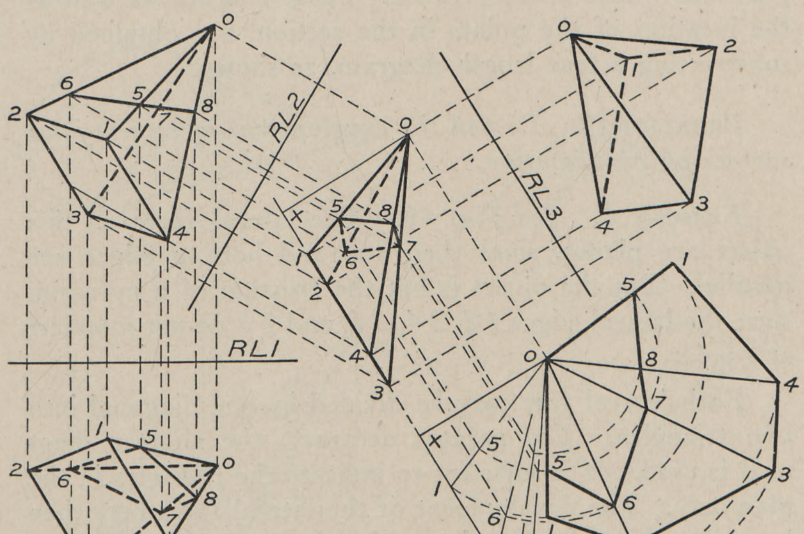
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Fig. 65.

Example 1. See Fig. 66. A right circular cone. The development of the curved portion is a sector of a circle, whose radius is the length of an element, and whose arc equals the perimeter of the circle of the base.

A section may be located by the distances of its points from the vertex *0*, and these distances may be found, as shown in the elevation, by projecting to the boundary element *0·1*.

Example 2. See Fig. 67. An oblique cone, with an elliptical base. This cone is shown with one view of the base a straight line, the other view in true size, since if not so given orginally, views of this kind can be obtained by projection. Assume elements of the surface, as shown; then develop the surface in the same manner as the pyramid, Fig. 65. The true length of each element is the hypothenuse of a right triangle whose base is the length of its plan and whose altitude is *0x*. These lengths, as well as the location of the points in the section, are obtained by constructing a true length diagram, as shown.

PROBLEM 19. *To find the development of an irregular surface by triangulation.*

Example 1. See Fig. 68. The lateral faces of this object are planes, since their top and bottom edges are parallel. But the object is not the frustum of a pyramid, since the lateral edges *1·2, 3·4, 5·6,* and *7·8* do not intersect at a point.

Each lateral face may be divided by one diagonal into two triangles. For greater accuracy, the shorter diagonal is used. It is sufficient to indicate the triangles in the plan only. The development of the lateral faces may then be built up by plotting these triangles in order, using the true lengths of the sides. Some of the true lengths are given in the plan. The others are found by constructing a true length diagram, as shown at the left of the elevation.

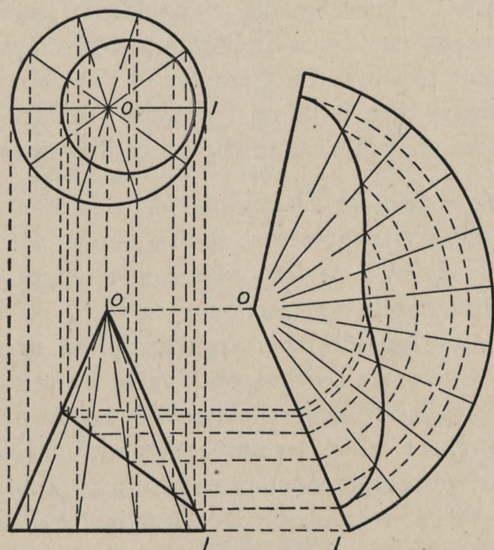
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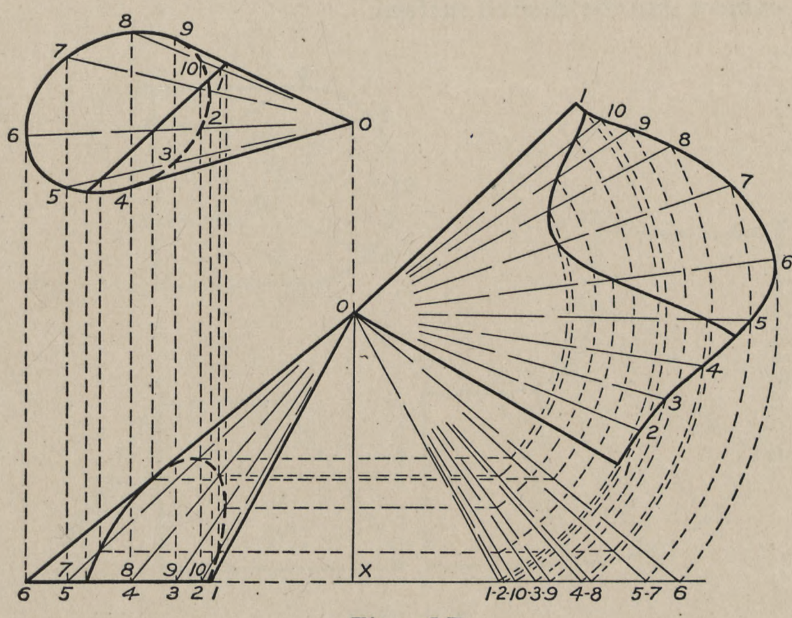
Fig. 66.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1-2-10-3-9
4-8
5-7
6

Fig. 67.

Example 2. See Fig. 69. The lateral surface, partly plane and partly curved, is divided into convenient triangles. It is sufficient to show the triangles in the plan only. The development is then built up from the true lengths of the sides of these triangles, as in the preceding example.

53. *Approximate Developments.* In the method of development by triangulation, shown above, the actual surface is replaced by a series of planes, triangular in shape, which approximately represent the surface. If the surface is actually developable, this approximation gives a result not quite so accurate as that obtainable by other methods, if such are available; nevertheless, it is a true development. But for surfaces which are not actually developable, an approximate development may often be obtained by a skilful placing of the triangles. Development by triangulation is of frequent occurrence in sheet metal work, where more or less stretch of the metal will convert an approximate development into the desired surface.

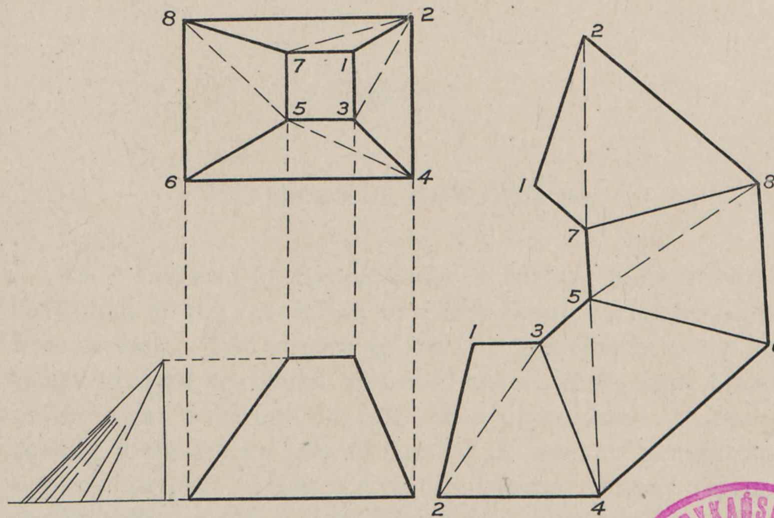
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Fig. 68.

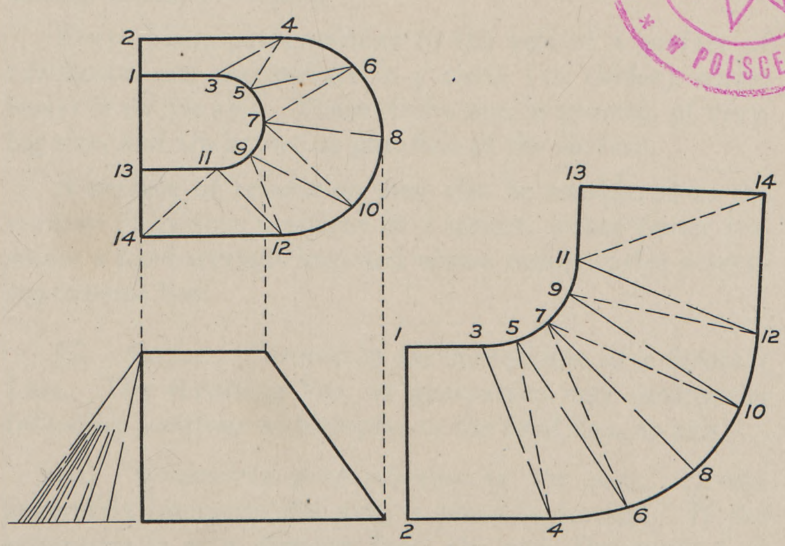
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Fig. 69.

Chapter 5

Surfaces of Revolution

54. *Surfaces of Revolution.* A surface of revolution is formed by the revolution of a line about a fixed straight line as axis. The revolving line, or generatrix, may be either straight or curved, and need not lie in the same plane as the axis; need not, in fact, be a plane curve. Consequently there are an infinite number of ways of generating a given surface. However, in the absence of any reason to the contrary, the surface is usually assumed to be generated by its meridian section, which is the line, straight or curved, cut from the surface by any plane which contains the axis. All meridian sections, or meridians, as they are usually called, are equal.

Every plane perpendicular to the axis of a surface of revolution cuts the surface in a circle (or circles) whose center is on the axis. These circles are, in general, of varying size, and are known as parallels of the surface.

A surface of revolution may also be considered as the envelop of various positions of a sphere, whose center traverses a fixed straight line, and whose radius varies according to some law.

55. *Surfaces Formed by the Revolution of a Straight Line.* The revolving line, or generatrix, may occupy the following positions with respect to the fixed line, or axis.

(*a*) *Generatrix perpendicular to the axis.* If the generatrix intersects the axis, a plane is formed. If the generatrix is at a distance from the axis, the result is a plane with a circular hole in it. These are degenerate

cases of some of the following surfaces, and need not be considered further.

(*b*) *Generatrix parallel to the axis,* consequently always in the same plane as the axis. The surface formed is the cylinder of revolution, or right circular cylinder. This is a developable surface.

(*c*) *Generatrix oblique to the axis,* and intersecting the axis, consequently always in the same plane as the axis. The surface formed is the cone of revolution, or right circular cone. This is also a developable surface.

(*d*) *Generatrix oblique to the axis,* but not intersecting the axis, consequently not in the same plane as the axis. The surface formed is a warped surface known as the hyperboloid of revolution of one nappe. The name is derived from the fact that the surface may also be formed by the revolution of a hyperbola about its minor, or conjugate, axis. This surface is not developable.

Since these are the only possible relative positions of a rectilinear generatrix and an axis, the above are the only surfaces of revolution which are ruled surfaces. The cylinder and cone of revolution are the only surfaces of revolution which are developable.

The cylinder of revolution may be considered as the envelop of a moving sphere, whose center traverses a fixed straight line, and whose diameter remains constant. It may also be considered as generated by a moving line always tangent to a fixed sphere and parallel to a given fixed line.

The cone of revolution may be considered as the envelop of a moving sphere, whose center traverses at uniform speed a fixed straight line, and whose diameter increases or decreases at a uniform rate. It may also be considered as generated by a moving line always tangent to a fixed sphere and passing constantly through a fixed point outside the sphere.

56. *Double Curved Surfaces of Revolution.* The simplest double curved surface of revolution is the sphere. As a surface of revolution, it may be derived by revolving a semi-circle about its chord. The sphere is unique in possessing an infinite number of axes; any diameter may be taken as an axis.

Every orthographic projection of a sphere is a circle, whose diameter is equal to that of the sphere. Hence to project a sphere in any view, it is only necessary to project its center, and then use this point as the center of a circle of the known diameter. On this property depends much of the usefulness of the sphere in determining the projections of other surfaces.

Double curved surfaces of revolution may be formed by the revolution of an ellipse about either axis, a parabola about its axis, or a hyperbola about its major, or real, axis. Also, by the revolution of any plane or space curve, which does not give one of the ruled forms previously described.

A torus is formed by the revolution of a circle about an axis situated in its plane, but outside the circle. This surface appears in elbows and bends in circular pipe, and in rings and chain links forged from circular rods. The convex portion of the surface, formed by the half of the circle farthest from the axis, also appears in architectural moldings at the base of columns. The surface may be considered as the envelop of a moving sphere of constant diameter, whose center traverses a fixed circle.

PROBLEM 20. *To pass planes tangent to a sphere through a given line.*

Let *o* be the center of the sphere, and let *ab* be the given line. In order that a solution may be obtained, the given line must not intersect the sphere.

Basic Solution. See Fig. 70. Let the line be given so that one of its projections is a point. Call this the first

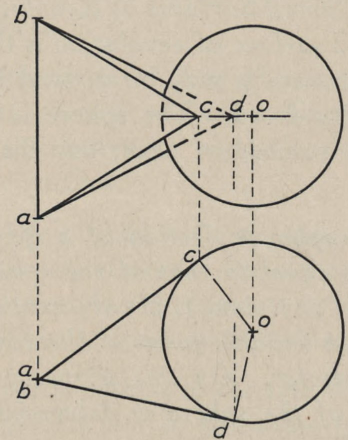


Fig. 70.

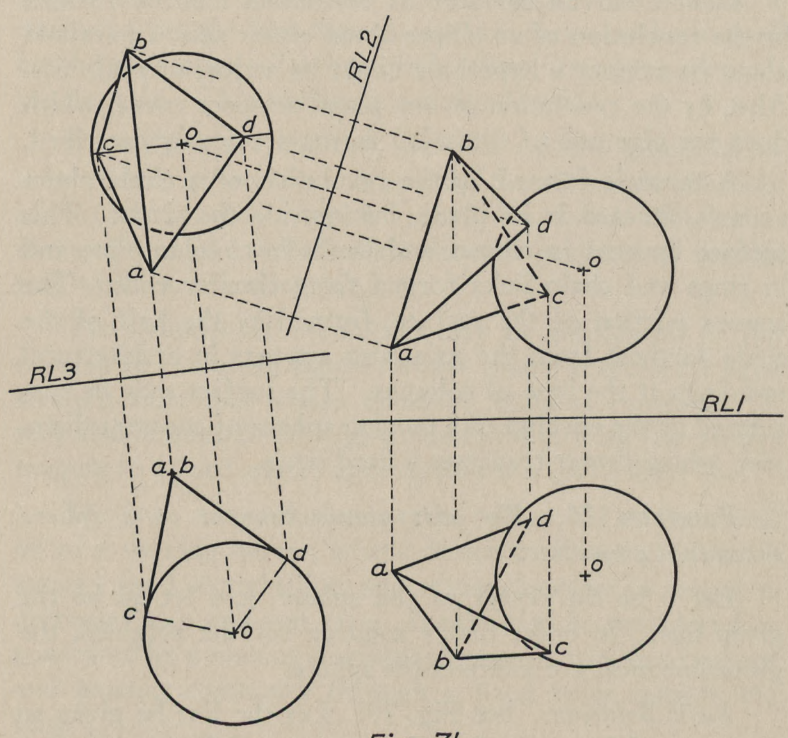


Fig. 71.

view. In this view, draw at once the edge views of the required planes, through *ab* tangent to the outline of the sphere.

Let it further be required to locate the points of tangency, *c* and *d*, of these planes, so that the planes are determined as the triangles *abc* and *abd*. The tangent points may be found by drawing the radii of the sphere which are perpendicular to the tangent planes. In the first view, from *o* draw perpendicular to the tangent planes; the intersections *c* and *d* are the points of tangency. These points lie on a great circle of the sphere whose plane is perpendicular to the line *ab*. In the second view, represent this plane by drawing a line through *o* perpendicular to *ab*. Project *c* and *d* to this line, and join them to the points *a* and *b* on the given line.

General Case. See Fig. 71. Find a third view of the line and sphere, in which the line projects in true length. Find a fourth view, in which the line projects as a point. In this view find the edge views of the tangent planes and the points of tangency. Project back to the original views.

PROBLEM 21. *To pass a sphere through any four given points.*

Let *a, b, c,* and *d* be the given points. In order that there may be a solution, the point *d* must not be in the same plane as points *a, b,* and *c*. The sphere will become known as soon as its center, *o*, and radius are determined.

If two parallel planes intersect a sphere, each section is a circle, and the line joining the centers of these circles lies on a diameter of the sphere. Therefore, begin by passing a plane through any three of the given points, as *a, b,* and *c*, and a parallel plane through the fourth point, *d*. The manner of doing this depends on the position of the given points.

Basic Solution. See Fig. 72. Let the three points *a, b,* and *c* be so given that the plane *abc* projects edgewise in the first view, and normally as the triangle *abc* in the second view. The parallel plane through *d* is drawn as a straight line in the first view, and is known to show normally in the second view.

In the second view, since the triangle *abc* is in true size, a circle passing through these points is also in true size. Find the center, *o*, of this circle. This circle lies on the surface of the required sphere, and the diameter *1·2*, projected to the first view, gives two points in the outline of the sphere.

In the second view, with the point *o* already determined as center, draw a circle passing through the point *d*. Project the diameter *3·4* to the first view, giving two more points in the outline of the sphere.

The sphere may now be drawn in the first view, as a circle passing through the points *1, 2, 3,* and *4*. With the radius thus determined, draw the second view of the sphere, using *o* as center.

General Case. See Fig. 73. Draw the projections of the plane *abc* as a triangle in the given views. The parallel plane through *d* need not be represented in these views in any way.

Make a third view so as to show the plane *abc* edgewise as a line. Project the point *d*, and represent the parallel plane by another line.

Make a fourth view so as to show the plane *abc* in true size, and project the point *d* to this view.

The solution is now effected in the third and fourth views as in the basic solution. As soon as two projections of the center of the sphere and its radius are obtained in these views, carry the results back to the original views.

PROBLEM 22. *To project a torus from any oblique angle.*

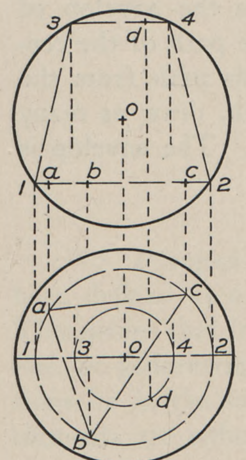


Fig. 72.

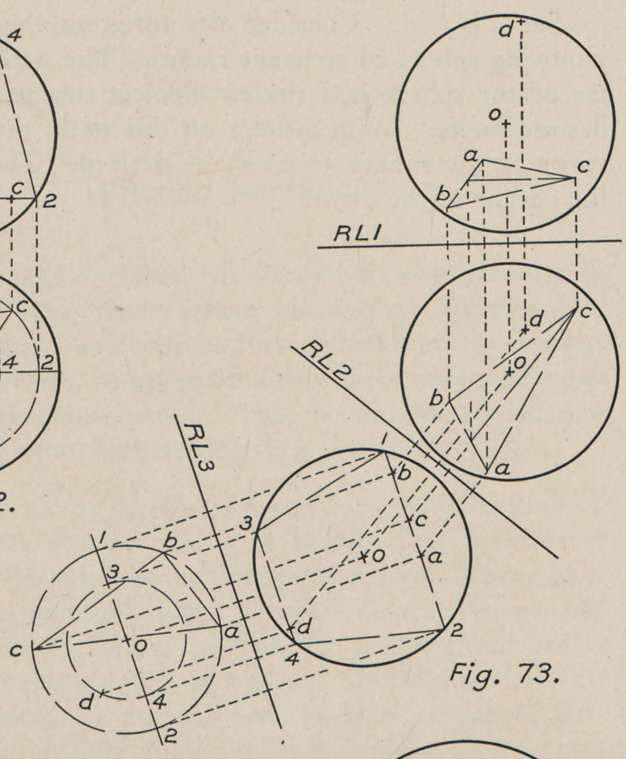


Fig. 73.

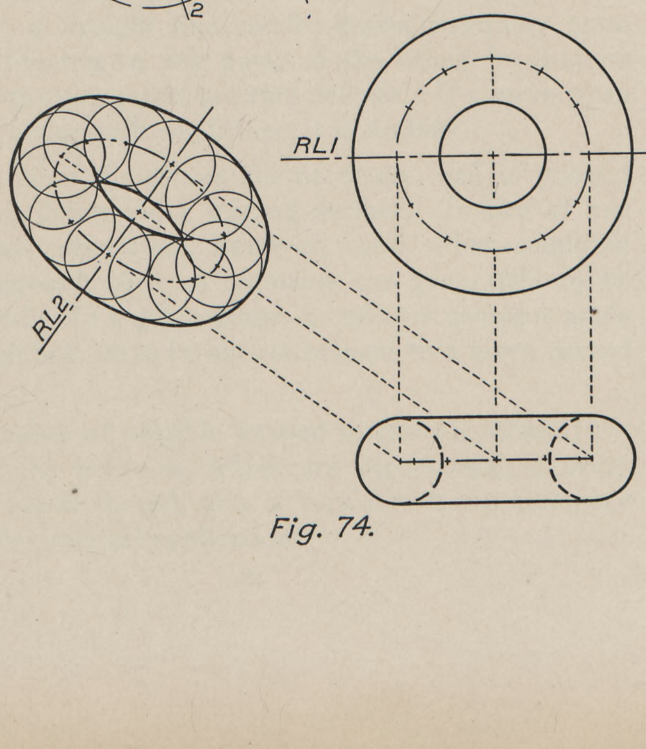


Fig. 74.

See Fig. 74. Consider the torus as the envelop of a moving sphere of constant radius. The path of the center of the sphere is a circle. Project this path from the desired angle. With centers on this path, draw as many views of the sphere (circles) as desired. The envelop is the outline of the torus.

Chapter 6

Warped Surfaces

57. *Warped Surfaces.* A warped surface has already been defined as a ruled surface which is not developable. This means that it is a surface formed by a moving straight line, or generatrix, so guided that no two consecutive positions lie in the same plane. That is, no two consecutive elements are either intersecting or parallel.

58. *Generation of Warped Surfaces.* One method of controlling the moving line is to require that it intersect certain fixed lines, known as linear directrices. These lines may be either straight, plane curves, or space curves, but must evidently be chosen so that the resulting surface is not a developable surface or a plane. It follows that two linear directrices are not sufficient to define a warped surface. For, a straight line can be drawn from any point of either directrix to any point of the other, so that no position of the generatrix becomes definite. If a surface can be formed under such conditions, it is a plane.

Three linear directrices are necessary, and in general sufficient, to determine a warped surface. In lieu of one of the linear directrices, however, some other condition may be imposed, such as requiring the generatrix to be always parallel to a given plane, to make a constant angle with a given line, or to be always tangent to a given curved surface.

The number of possible warped surfaces is infinite; but outside of the helicoids, which are represented in every screw and screw thread, only a very few forms have any considerable practical application.

59. *Singly and Doubly Ruled Surfaces.* Let a warped surface be generated with three straight lines as linear directrices. Take any three positions of the generatrix as a second set of linear directrices. The same surface may then be formed by the motion of a second generatrix, of which the original directrices are three positions. Hence, through any point of the surface two distinct rectilinear elements can be drawn, one corresponding to each method of generation. Such a surface, containing two distinct sets or systems of elements, is doubly ruled. An important feature of a doubly ruled surface is that every element of either system intersects every element of the other system.

All other warped surfaces are singly ruled; that is, through any point in the surface but one rectilinear element can, in general, be drawn.

60. *Representation of Warped Surfaces.* A straight line is indefinite in extent, so that a surface formed by one is also indefinite in extent. Hence only a limited portion of a warped surface can be shown. This is usually accomplished by giving the directrices, and a number of elements contained in the part represented.

61. *The Hyperboloids.* All doubly ruled surfaces may be classed under the general name of hyperboloids. They may then be divided into elliptical, circular, and parabolic forms. All of these forms possess an axis of symmetry.

In the elliptical hyperboloid, every plane section perpendicular to the axis is an ellipse, and every plane section containing the axis is a hyperbola. It is the general form of surface which results when any three straight lines, chosen at random, are used as directrices. This form is of but limited application.

In the circular form, every plane section perpendicular to the axis is a circle, and every plane section containing the axis is an equal hyperbola. It is therefore a surface of

revolution, and is most easily generated and described as such.

From either the elliptical or circular hyperboloid the usual conics, circle, ellipse, parabola, and hyperbola, may be obtained by suitably chosen cutting planes.

The parabolic form of the hyperboloid is usually called the hyperbolic paraboloid. It will be generated and described under this name. The plane sections of this form of the surface consist only of parabolas and hyperbolas.

62. *The Unparted Hyperboloid of Revolution.* The unparted hyperboloid of revolution may be formed by the revolution of a hyperbola about its conjugate axis, and is so called to distinguish it from the surface formed by revolving the hyperbola about its principal axis. The latter is not a warped surface.

Considered as a warped surface, the simplest method of generating the unparted hyperboloid of revolution is by revolving one straight line about another not in the same plane. Every point in the generating line describes a circle lying in a plane perpendicular to the axis. The point nearest to the axis describes the smallest circle, known as the circle of the gorge. Points which are equidistant from this point describe circles of the same size, equidistant from the circle of the gorge. A customary representation of the surface consists of the portion between two equal circles, giving a symmetrical projection as shown in Fig. 75. In this representation, the positions of the generating line, or elements of the surface, are tangent in the plan to the circle of the gorge. The hyperbolic outline in the elevation is the envelop of the elements.

In Fig. 75 the elements shown represent the positions of but one generating line. The surface, however, has been previously described as doubly ruled, and hence can be generated by either of two lines. This should be apparent from Fig. 76. Either of the lines B or C, which are sym-

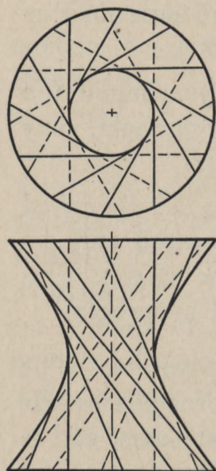


Fig. 75.

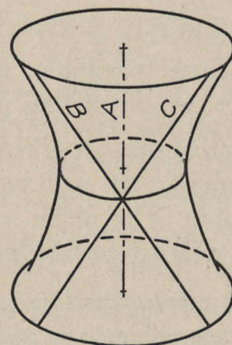


Fig. 76.

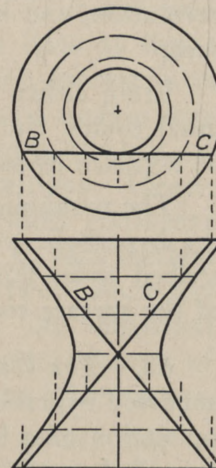


Fig. 77.

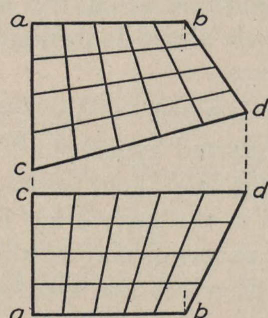


Fig. 79.

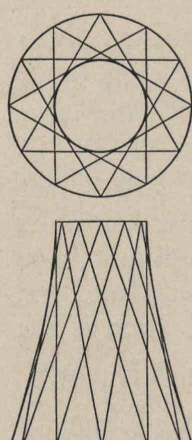


Fig. 78.

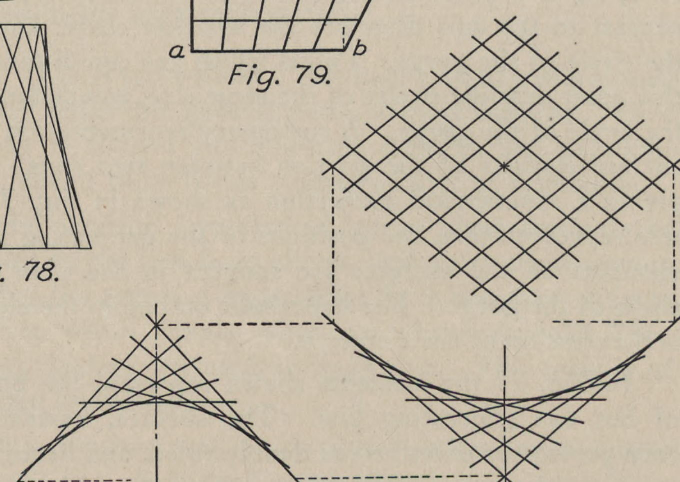


Fig. 80.

metrically placed with respect to the axis A, will generate the surface.

The surface may also be represented by drawing a number of parallels of the surface, that is, by drawing the circular paths of a number of chosen points on the generatrix. Such a representation is shown in Fig. 77. In this figure it should be evident that either line B or C will generate the surface, and that the elevations of these lines are the asymptotes to the hyperbolic outline.

63. *Applications.* Because of its double ruling, this surface, either in the circular form just described, or in the more general elliptical form, can be constructed as a lattice work of straight bars, fastened together at every point of intersection. Such a lattice was at one time used as a firing mast on warships. Since for stability the mast should be smallest at the top, only the portion below the gorge is used. See Fig. 78, which shows the circular form. If it is desired to use this lattice as a pier whose top is longer than it is wide, the elliptical form can be employed.

The hyperboloid of revolution is sometimes used as the pitch surface for skew gears, that is, for gears revolving on axes which are not in the same plane. See Problem 23, Corollary.

64. *The Hyperbolic Paraboloid.* In the elliptical and circular forms of the doubly ruled hyperboloid, the generatrix, after generating the complete surface, can be brought to its initial position. In the parabolic form no such return is possible within finite limits. In this form, planes passed through the axis cut the surface in parabolas, planes perpendicular to the axis cut hyperbolas. Hence the usual name of the surface, hyperbolic paraboloid.

65. *Generation.* The hyperbolic paraboloid may be generated by taking as linear directrices three straight lines all parallel to the same plane (no two lines parallel or inter-

secting). But since the surface is doubly ruled, the directrices of one generation becomes the elements of the other. Hence the surface may also be generated by using as linear directrices two elements of the first generation, and as a plane director the plane to which the original directrices are parallel. Whence, from symmetry, either set of elements may be obtained by using two rectilinear directrices and a plane director.

A more usual method of obtaining a portion of the surface is shown in Fig. 79. Let *a, b, c,* and *d* be four points not in the same plane. Divide the opposite lines *ac* and *bd* into the same number of equal parts, and connect the corresponding points of division. Similarly for the opposite lines *ab* and *cd*. The resulting lines are elements of a hyperbolic paraboloid. In this figure, it is evident that one set of elements is horizontal, and that any horizontal plane can be used as a plane director. The plane director for the other set of elements is not apparent.

A symmetrical projection of the surface is given in Fig. 80. In this position of the surface the axis and the parabolic outline are readily seen.

66. *Applications.* This surface can be used for the face of a concrete wall of varying slope (batter), since the form for pouring the concrete can readily be made of boards placed to correspond with the elements of the surface. It can be similarly used in a roof of varying slope (pitch), the rafters representing one set of elements, and the cover boards the other set. The surface has also been used in ship building, and as the pilot of locomotives.

67. *Tangent and Normal.* A normal at any point in a surface is a line perpendicular to the surface at that point. A tangent plane is perpendicular to the normal at the point of tangency, and coincides with the direction of the surface at that point. In a ruled surface, the tangent plane at any point contains the rectilinear element passing through the

point. In a warped surface the direction of the surface is constantly changing, so that, in general, no two points on the same element have the same tangent plane.

In doubly ruled surfaces, the tangent plane at any point is the plane determined by the two rectilinear elements, one of each system, which pass through the point. The normal, being perpendicular to this plane, is perpendicular to each of these elements.

In the case of a surface of revolution, the normal at any point in the surface intersects the axis.

68. *Tangent Hyperboloids of Revolution.* Let E (no figure) be any element of a hyperboloid of revolution, and A the axis of the surface. The normal at any point of E is perpendicular to E, and intersects A. Let all the normals be drawn along the entire length of the element E; then the surface formed by them will be a hyperbolic paraboloid. For it has the two linear directrices A and E; and since every element is perpendicular to E, it is parallel to any plane perpendicular to E, which can serve as plane director. This surface therefore contains a second system of elements, of which A and E are two. Let B be another element of this system. Consider B to be the axis, and E an element, of a second hyperboloid of revolution. The normals to this surface along the element E will be the same as for the first hyperboloid with axis A. Consequently the two hyperboloids will be tangent to each other along the entire length of the common element E. This analysis enables us to solve the following problem.

PROBLEM 23. *To construct two hyperboloids of revolution which shall be tangent to each other along a common element.*

See Fig. 81. Let A be the axis of one of the hyperboloids, and E the common element. It is required to find the axis, B, of the second hyperboloid.

Let *no* be the common perpendicular to both *A* and *E*, so that point *o* describes the circle of the gorge. Assume any other point, *s*, on *E*. Draw the normal *sr*, perpendicular to *E*; in the plan it shows perpendicular to *E*, and in the elevation passes through the point view of *A*. Produce the normal; on it assume any point *t*. From *t* draw the line *B*, perpendicular to and intersecting the line *no* produced. Then *B* may be taken as the axis of a second hyperboloid of revolution, tangent to the first one along the common element *E*.

The construction of the two hyperboloids is not given in Fig. 81, in order to show more clearly the relations between the axes *A* and *B*, the common element *E*, and the common normals *noq* and *rst*, all of which are elements of the hyperbolic paraboloid previously described.

COROLLARY. *Given the axes, to draw two equal hyperboloids of revolution tangent to each other along a common element.*

See Fig. 82. The axes being placed as shown, it is evident that the common element *E* bisects in the plan the angle between the axes *A* and *B*, and in the elevation bisects the distance between *A* and *B*. The hyperboloids are then obtained by revolving the line *E* about *A* and *B* in turn.

Note. Two equal hyperboloids of revolution, placed as above, can be used as the pitch surface for gears. For any two points in contact along the common element will revolve in circles of equal size, and consequently will have equal velocities. If the hyperboloids are unequal in size, as in the general case, the speed ratio does not remain constant along the length of the common element.

69. *Singly Ruled Surfaces.* While the number of doubly ruled surfaces is limited to the three forms just described, there is no limit to the number of singly ruled warped surfaces. Only a few forms, however, need be considered here.

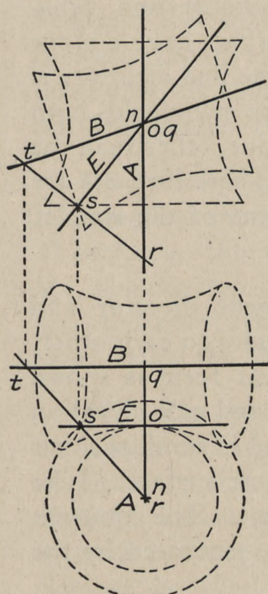


Fig. 81.

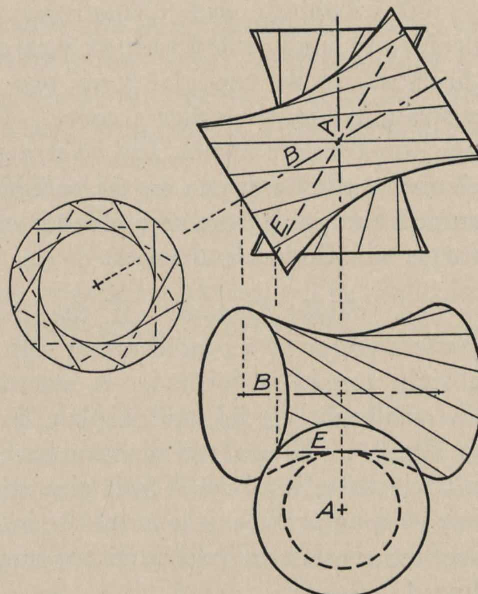


Fig. 82.

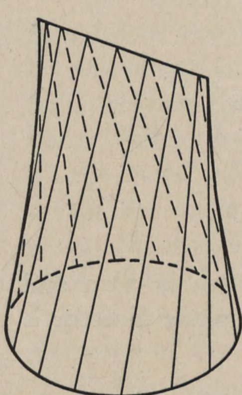


Fig. 83.

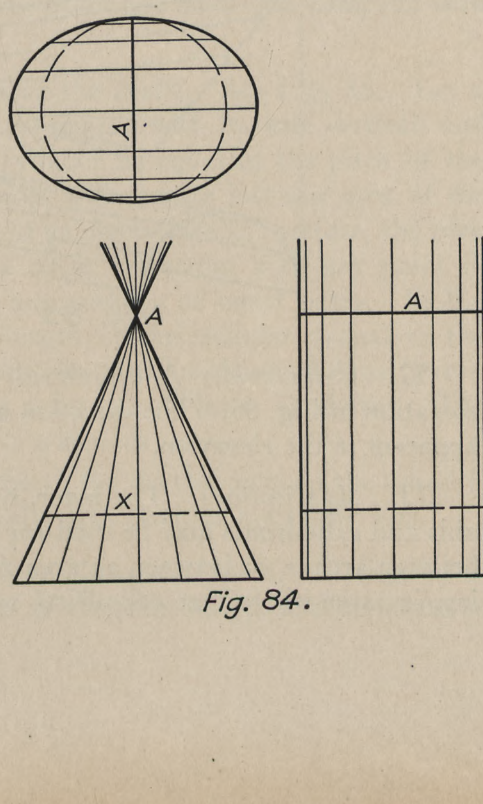


Fig. 84.

70. *Conoids and Cylindroids.* Conoids and cylindroids are singly ruled warped surfaces which have a plane director. In the conoidal form, one of the line directors is a straight line, the other a curve. In the cylindroid, both line directors are curves, and no straight line, other than the elements, can be drawn on the surface. These surfaces are named because of the resemblance of some of the simpler forms to cylinders and cones.

71. *Right Conoid.* If the rectilinear director and plane director of a conoid are at right angles to each other, a right conoid is formed. A symmetrical form is shown pictorially in Fig. 83, and in plan, front, and side elevation in Fig. 84. The curved director here used is an ellipse, but a circle lying in plane X will give the same surface. Like any warped surface, it is indefinite in extent. The complete surface consists of two parts, or nappes, separated by the line A.

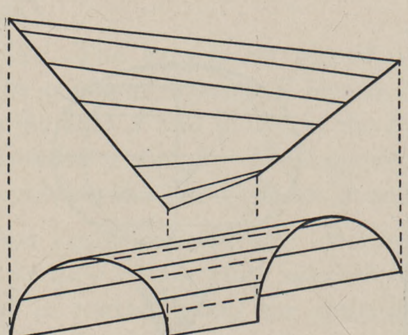


Fig. 85.

72. *Cylindroid.* A cylindroid is shown in plan and elevation in Fig. 85. The direction of the plane director is apparent in the elevation.

73. *Applications.* The inner, or concave, side of conoids and cylindroids may be used for ceilings, as transitions between arches, or between an arch and a flat or plane ceiling, in cases where true cylindrical or conical forms cannot

be used. The surfaces may also be employed for transitions between round and rectangular forms, in pipes, tops of manholes, etc., when made of cast metal. But as they are not developable, they are not available in sheet metal work.

74. *Helicoids*. Helicoids, on account of their numerous practical applications, form a large and important class of surfaces. There is one developable form; all the others are non-developable or warped surfaces. The helicoids are all derived from a space curve, or curve of three dimensions, called the helix.

75. *The Helix*. For present purposes, the helix will be defined as a curve drawn on the surface of a cylinder of revolution, making a constant angle with the elements. The distance measured along any element of the cylinder, between two successive points in which the helix crosses this element, is called the pitch of the helix. The pitch consequently represents the linear advance of a point moving along the helix after the point has made one complete turn around the axis.

76. *Construction of the Helix*. See Fig. 86. Let a cylinder of revolution be placed with its axis vertical, and let the length of the cylinder be equal to the pitch of the helix, so that the cylinder will contain just one turn of the helix. Since the curve is on the cylindrical surface, the plan of the helix will be a circle. Starting with any point *0*, divide the circle into any number of equal parts. In the elevation, divide the length of the cylinder (equal to the pitch of the helix) into the same number of equal parts. Project corresponding points of division, as shown. The elevation of the helix is a sinuous curve.

In Fig. 86 is also shown the development of the cylindrical surface, with the helix appearing as a straight line.

77. *Right and Left Handed Helices*. In Fig. 87 a cylinder has been taken of the same length and diameter as

in Fig. 86, and a helix drawn in the opposite direction. There is also given in Fig. 87 a development of the concave, or inner side, of the cylindrical surface, as in Fig. 86, showing the cylinder as a rectangle. But in the two developments the helix is not represented by the same diagonal of

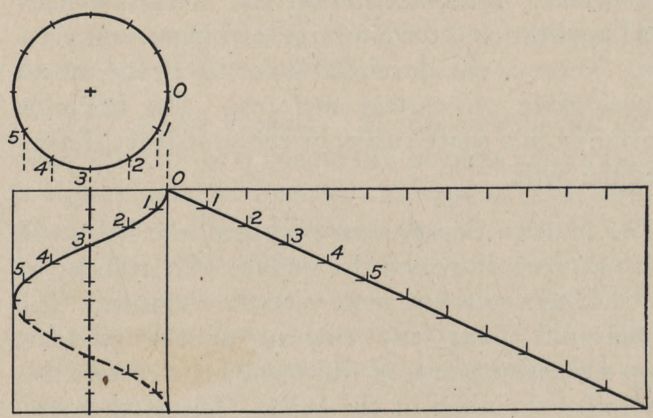


Fig. 86.

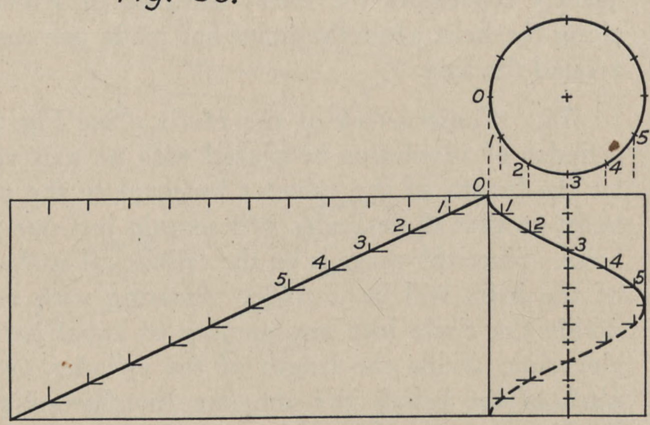


Fig. 87.

the rectangle. The two helices are different; although drawn on equal cylinders and with the same pitch, they cannot be made to coincide. To distinguish them, the helix in Fig. 86 is called right handed, the one in Fig. 87 is left handed. This is an arbitrary, but universal, convention.

To distinguish between a right and a left handed helix, look at the helix along its axis, that is, at the view which is a circle. Follow the path of a point which travels along the helix away from the observer. If the point appears to rotate right handed, that is, clockwise, the helix is right handed. If the rotation appears left handed, or counter-clockwise, from this point of view, the helix is left handed.

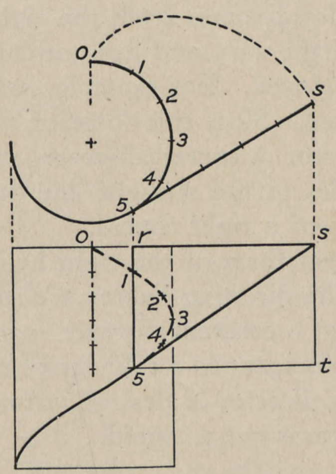


Fig. 88.

78. *A Line Tangent to a Helix.* Since a helix develops into a straight line, a tangent to a helix may be drawn by arranging the development so as to stop at the point where the tangent is desired. In Fig. 88 the line *5s* is tangent to the helix at point *5*. In the plan, *5s* is tangent to the circle, and the length from *5* to *s* equals the length of the arc from *5* to *0*. In the elevation, point *s* is located in the same horizontal plane as *0* by projecting from the plan. Note that in the elevation the rectangle *5rst* is not the true development of the cylinder between elements *0* and *5*, but is the projection (elevation) of this development, so that the elevation, *5s*, of the developed helix is tangent to the elevation of the helix at point *5*.

79. *The Right Helicoid.* The right helicoid has two helical directrices and a plane director, the helices having the same pitch and a common axis, and the plane being perpendicular to this axis. In its simplest form, Fig. 89, the elements intersect the axis, which may therefore be substituted for one of the directing helices. The surface may consequently be represented by elements intersecting the helix and its axis, at right angles to the axis. The designation of right helicoid comes from the right angle between the elements and the axis, and has nothing to do with the form of helix employed. For example, in Fig. 89, the helices are left handed. Since this form of the helicoid has a straight line director, a curved director, and a plane director at right angles to the straight line, it falls under the general definition of a right conoid.

A more general form of the right helicoid is shown in Fig. 90. Here the directing helices are so placed that the elements remain at a constant distance from the axis. The elements are thus tangent to a cylinder of revolution, which may be used as a director if desired instead of one of the helices. This form is not a conoid.

80. *Extent of the Surface.* The directing helices are indefinite in length, as is also the generating line. The surface is therefore of indefinite extent, consisting of an infinite number of turns. It is, however, divided into two equal parts, or nappes. If the elements intersect the axis, as in Fig. 89, the axis is the line of separation between the nappes. In the form of Fig. 90 this line is a helix lying in the cylindrical surface to which all the elements of the helicoid are tangent.

Since the elements of any right helicoid are all parallel to the same plane, no two will intersect, however far produced. Hence the surface does not intersect itself at any point.

81. *The Oblique Helicoid.* The oblique helicoid differs from the right helicoid in that the elements make a

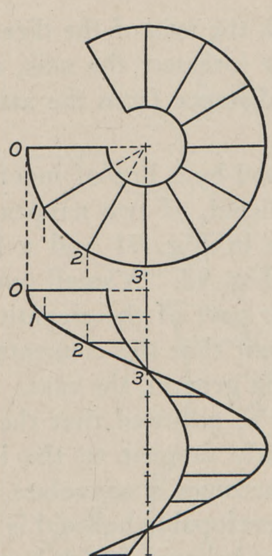


Fig. 89.

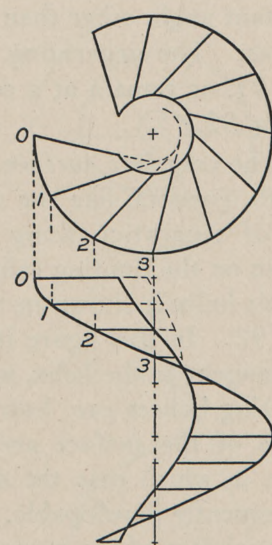


Fig. 90.

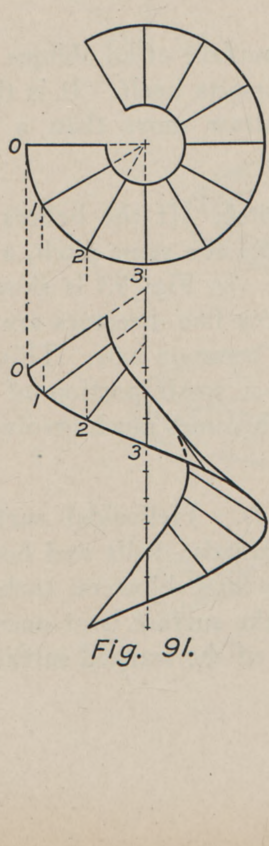


Fig. 91.

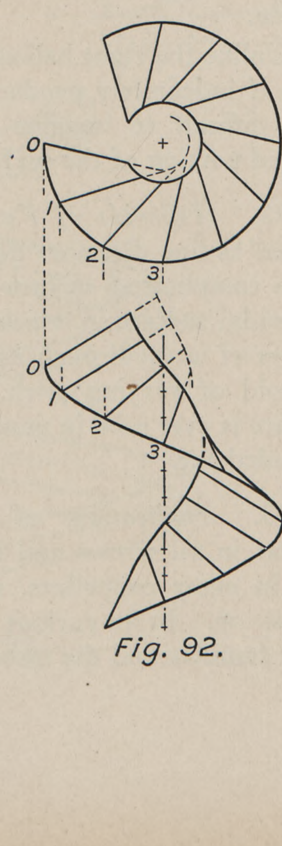


Fig. 92.

constant angle other than 90° with the axis of the directing helices. The generating line may intersect the axis, as in Fig. 91, or remain at a constant distance from the axis, as in Fig. 92.

The complete surface, generated by a line of indefinite length, consists, like the right helicoid, of two nappes, the line of separation being the axis in Fig. 91 and a helix drawn on the limiting cylinder in Fig. 92. A small portion of this helix is shown in the upper part of the elevation in Fig. 92. In this figure it is evident that the elements are not tangent to the helix, which is, in general, the case. The directing helices can, however, be so adjusted that the elements of the surface are constantly tangent to the inner helix, in which case the surface becomes a convolute, and consequently developable. The developable helicoid is thus a special case of the general oblique helicoid, which in every other case is a warped surface, consequently not developable.

Unlike the right helicoid, the surface of an oblique helicoid, if indefinitely produced, intersects itself. It is therefore difficult to imagine or to draw more than a very limited portion of the surface.

82. *Helicoids of Varying Pitch.* If the helices employed as line directors do not have the same pitch, a still more complicated surface results. In Fig. 93 is shown a helicoidal surface in which the three line directors are two helices of unequal pitch and their common axis. This is a helicoid of varying pitch. Only a small portion of this surface is ever used in practice; it becomes much involved if extended far.

83. *Applications of Helicoids.* Helicoidal surfaces appear in all screws and screw threads, drills and borers, and in many propellers, rotating fans, blowers, turbines, and so on. In its various forms the surface is at once the most familiar and the most useful of the warped surfaces.

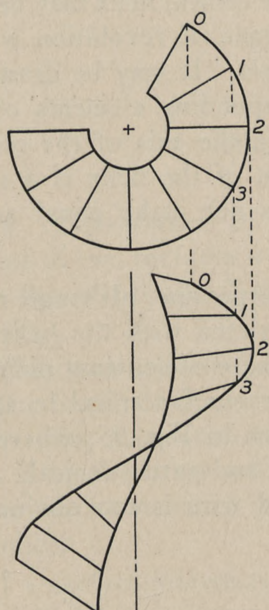


Fig. 93.

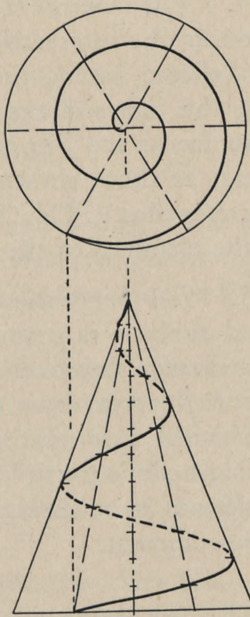


Fig. 94.

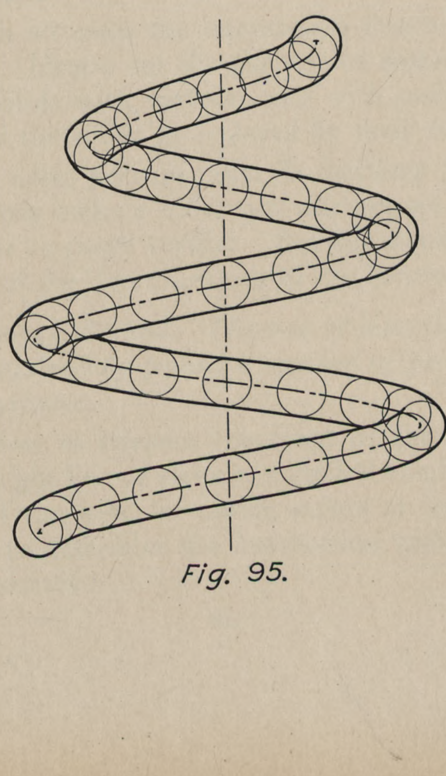


Fig. 95.

84. *The Conical Helix.* The conical helix may be considered as the intersection of a cone of revolution with a right helicoid having the same axis. It may be drawn by noting the intersections of corresponding elements of the two surfaces; see Fig. 94. When the axis of the cone is vertical, as there shown, the plan of the helix is a spiral of Archimedes. This helix does not make equal angles with the elements of the cone.

85. *The Serpentine.* The serpentine, although not a warped surface, is naturally associated with the helicoids. The surface is the envelop of a sphere of constant diameter whose center traverses a helix or similar spiral in space. One view of a serpentine is shown in Fig. 95, where the directing helix is partly cylindrical and partly conical. The resemblance to a spring of round wire is too obvious to require comment.

Chapter 7

Intersections

86. *Intersections of Surfaces and Solids.* The intersections of two surfaces is a line or lines, straight or curved, which passes through all the points that are common to both surfaces.

When two solids intersect, their surfaces intersect. The line (or lines) of intersection, as traced on the surface of either solid, forms the boundary of the hole which is cut from it by the other solid. This line, actually the intersection of two surfaces, is usually called the intersection of the solids.

If a surface intersects a solid, the boundary line of the section is the intersection of the given surface with the surface of the solid.

So that in all cases the intersection desired is that of two surfaces. Hence, for the purpose of finding the intersections of solids with surfaces and with each other, we shall consider the solids as replaced by their surfaces.

We may note, perhaps, that in deciding questions of visibility, it may make a difference whether we are dealing with a solid or an empty surface. But this matter is entirely independent of the determination of the intersection.

87. *General Method. The line of intersection of two surfaces is obtained by finding a number of points common to the two surfaces.*

If we know in advance the form of the intersection, such as a straight line or a circle, a limited number of points is enough. Otherwise, we obtain points at intervals sufficiently close to determine the intersection with the desired degree of accuracy.

88. *The Intersection of Two Planes.* Since the intersection of two planes is known to be a straight, two points will be sufficient to determine it. We may therefore draw two straight lines in one of the planes, then find and join the two points in which these lines intersect the other plane. *The solution of this problem therefore depends on the problem of finding the intersection of a straight line and a plane.*

89. *The Intersection of a Straight Line and a Plane.* One method of finding the intersection of a straight line and a plane is to make an additional view, showing the plane edgewise (Prob. 7). However, when this problem is used as an auxiliary problem in finding the intersection of surfaces, the following method of solution, which does not require an additional view, will generally give quicker results.

LEMMA 4. *To find the point of intersection of a straight line and a plane without making an additional view.*

See Fig. 96. Let *A* be the given line, and *1·2·3* the given plane.

CONSTRUCTION. In one of the given views, say the plan, assume a line *K*, whose plan coincides with the plan of the given line *A*.

Let *K* represent a line in the given plane *1·2·3*. On this basis find the elevation of line *K* (Lemma 1).

In the elevation, mark the point *c*, where the lines *A* and *K* intersect. Project this point to the plan.

Point *c* is the point in which the given line intersects the given plane.

Proof. The lines *A* and *K* lie in a plane, namely, the plane which is seen edgewise in plan as the line *AK*. Therefore, if the elevations of *A* and *K* intersect, the lines intersect. Consider the point of intersection, *c*, of the lines *A* and *K*. First, it is a point in the given line *A*. Second, it

is a point in line *K*. But line *K* lies in the given plane *1·2·3,* so that any point of *K* lies in this plane. Hence point *c* lies in the plane *1·2·3*. That is, point *c* lies in both line *A* and plane *1·2·3*, consequently it must be the point of intersection of the line and the plane.

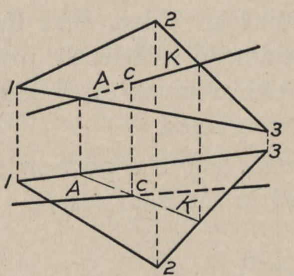


Fig. 96.

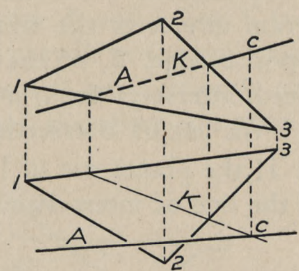


Fig. 97.

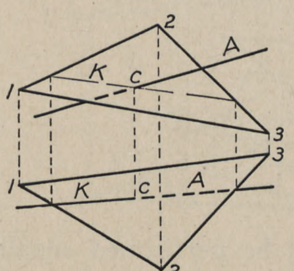


Fig. 98.

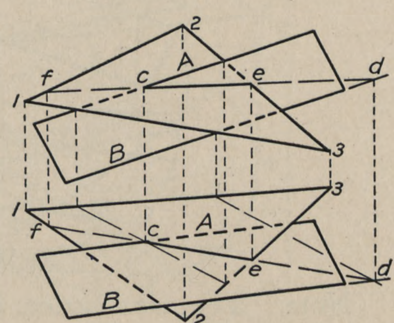


Fig. 99.

Note 1. If the lines *A* and *K* are found to be parallel, the given line is parallel to the given plane.

Note 2. With a plane of limited extent, as in the triangular form given, it does not follow that the point *c* will necessarily fall within the boundaries of the plane. The construction merely gives the point in which a line, sufficiently produced, intersects a plane of indefinite extent. See Fig. 97.

Note 3. It should be evident that the construction can be performed equally well by assuming lines *A* and *K* coincident in the elevation. See Fig. 98.

PROBLEM 24. *To find the line of intersection of two planes.*

Let the planes be given as in Fig. 99, one a triangle, the other a parallelogram.

By Lemma 4 we find that the edge *A* of the parallelogram intersects the triangle at point *c*. Also, that the opposite edge, *B*, of the parallelogram, if sufficiently produced, intersects the triangle produced at point *d*. Hence *cd* is the line of intersection of the two planes.

If the planes are of limited extent, only the portion *ce* of the line of intersection is retained in the final result.

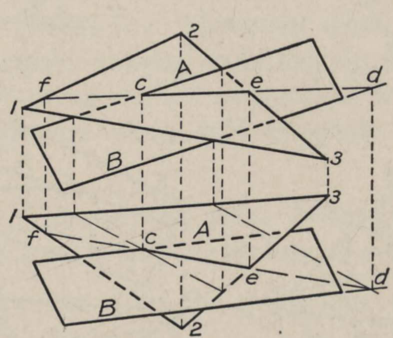


Fig. 99.

Note. The construction could be performed equally well by finding the points in which the edges of the triangle intersect the plane of the parallelogram. Thus, *2·3* intersects the parallelogram at point *e*, and *2·1* at point *f* (construction not shown). Or, any two of the points *c, d, e,* and *f* will determine the line of intersection.

PROBLEM 25. *To find the intersection of a ruled surface and a plane.*

Draw straight line (elements) on the ruled surface. Find the points in which these lines intersect the given plane (Lemma 4), and connect the points in order.

Note. This method applies to the cone, cylinder, convolute, or any warped surface. It is illustrated in Fig. 100,

the surface there used being an oblique (elliptical) cone. In this figure, the visibility has been determined on the basis that the plane is not removed after the cut is made.

90. *The Use of Sectioning Planes.* Let A and B be any two surfaces. Suppose a plane S to be passed so as to cut a line J, straight or curved, from A, and a line K from B. If J and K intersect, which, as they lie in the same plane they may do, the point (or points) of intersection is common to both the given surfaces, and therefore is a point in their line of intersection.

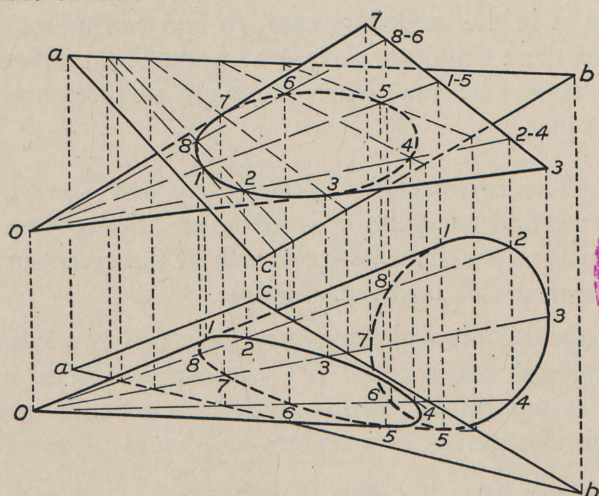


Fig. 100.

This is a general method of finding points in any case of the intersection of surfaces. In its practical application, two conditions are usually observed. *First, the sectioning plane is located in a view where the plane can be drawn edgewise, as a straight line.* This may necessitate the use of one, and sometimes two, additional views. *Second, the sections cut from the given surfaces are, when possible, either straight lines or circles.*

91. *Sectioning Planes Cutting Straight Lines.* Straight lines can be drawn only on ruled surfaces. If one of two

intersecting surfaces is any ruled surface, and the other a plane, any plane containing an element of the ruled surface will cut the given plane in a straight line, whose intersection with the element is a point in the required intersection of the surfaces. This gives another method of interpreting the construction of Problem 25. In Fig. 100, the edge views of the sectioning planes may be considered as coincident with the elevations of the elements of the curved surface.

If both of the ruled surfaces are curved, it is not always possible, as in the preceding case, to find sectioning planes which cut them simultaneously in straight lines. However, this can always be done when the given surfaces are cylinders or cones.

92. *The Intersections of Cylinders and Cones.* The general method of solution consists in passing sectioning planes so as to cut elements from each of the given surfaces. As just stated, this is always possible.

The three cases which may arise are (*a*) two cylinders; (*b*) a cylinder and a cone; (*c*) two cones. These will be treated as separate problems. Also, in general, we shall not confine ourselves to cylinders and cones of revolution.

PROBLEM 26. *To find the intersection of two cylinders.*

See Fig. 101. Obtain an end view of one (either) of the cylinders. In the given figure, the better cylinder to use is *A*, since its end view can be obtained by but one additional view. In the third view, any plane seen edgewise which intersects *A* will cut elements from this cylinder. To cut elements also from cylinder *B*, let the planes be taken parallel to the axis of *B*. A typical plane is *Q*. This cuts elements *E* and *F* from *A*, and elements *J* and *K* from *B*. These elements intersect in four points, which are points in the required intersection.

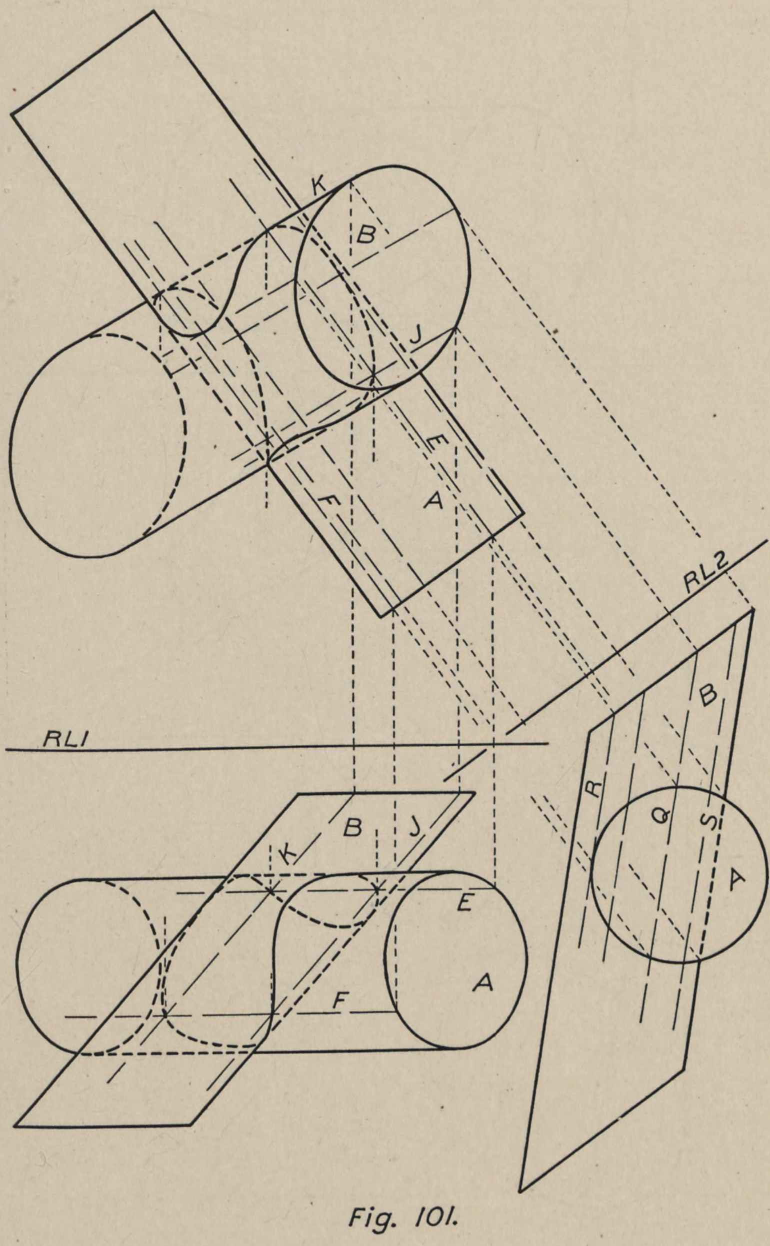


Fig. 101.

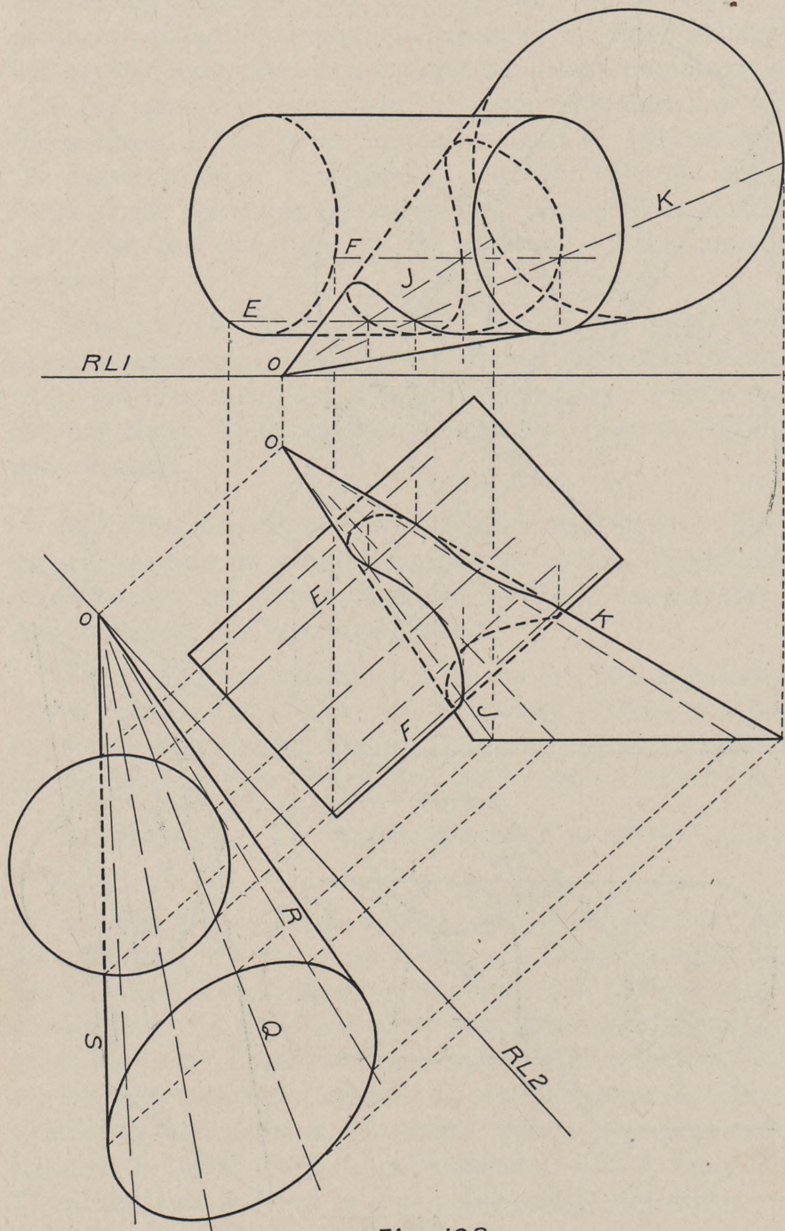


Fig. 102.

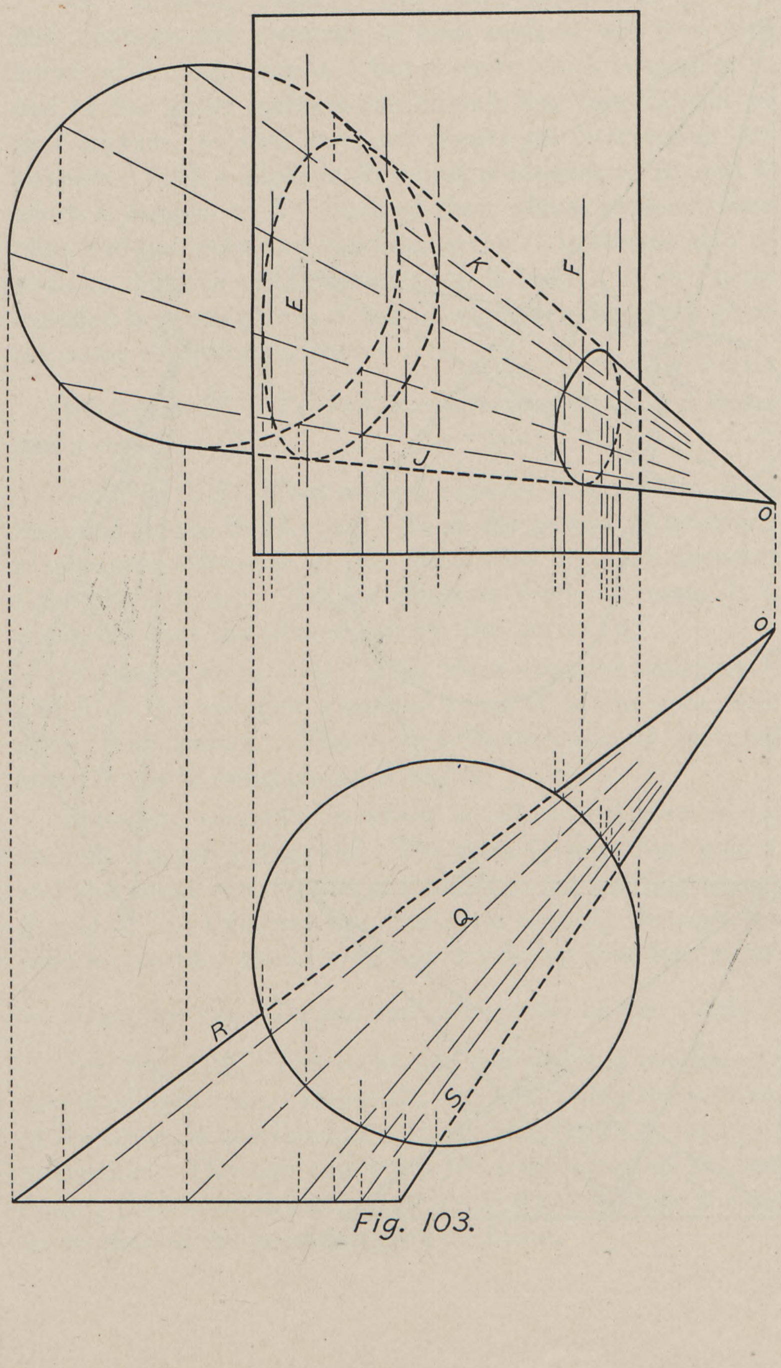


Fig. 103.

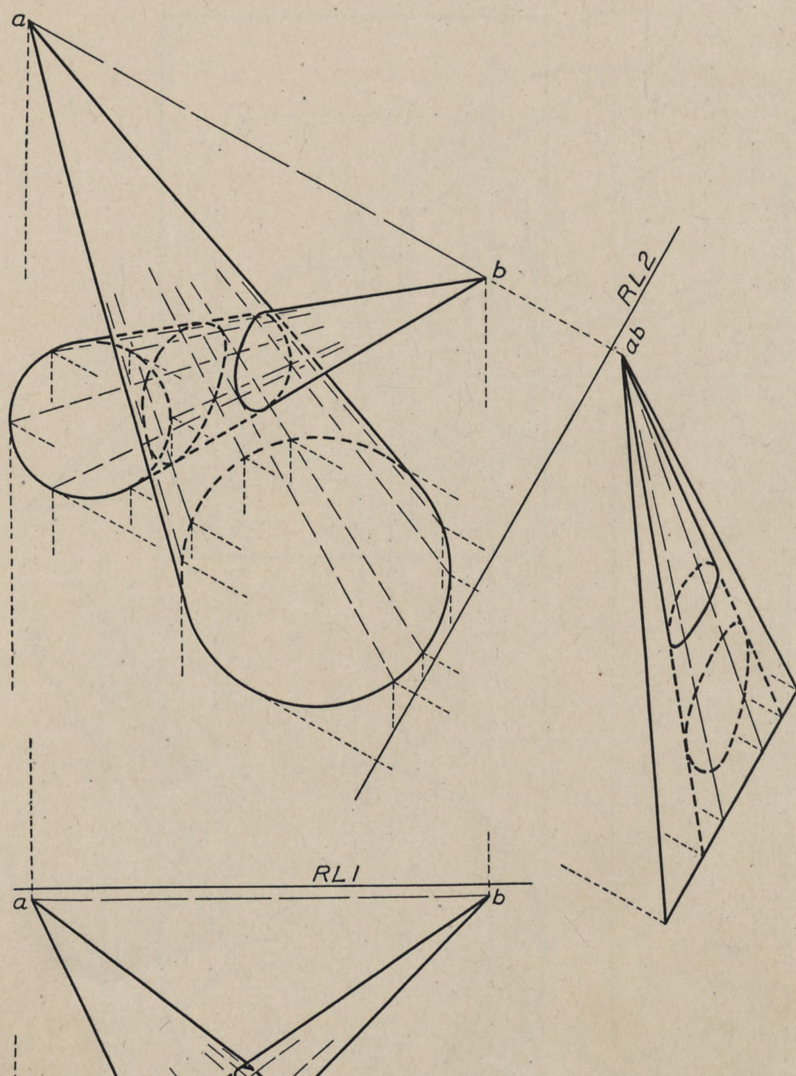


Fig. 104.

93. *Limiting Planes.* Any plane which, like Q, Fig. 101, contains two elements of each surface, will give four points of the intersection. But a plane which is tangent to one of the given surfaces can contain but one element of that surface, so that but two points of intersection are possible. Such a plane is R, which is tangent to A, and S, which is tangent to B. Each of these planes evidently contains but two points of the intersection. It should also be obvious that the entire intersection is limited to the space included between planes R and S, which are therefore given the name of limiting planes.

PROBLEM 27. *To find the intersection of a cylinder and a cone.*

See Fig. 102. Find an end view of the cylinder, here obtained in the third view. As in the preceding problem, planes seen edgewise in the third view will cut elements from the cylinder. To cut elements from the cone, it is necessary that the planes contain the vertex, o.

A typical plane is Q. This plane contains elements E and F of the cylinder, elements J and K of the cone, and gives four points. The entire intersection is included between the limiting planes R and S.

A second example, in which an additional view is not needed, is given in Fig. 103. The plane Q gives four points, and the entire intersection lies between the limiting planes R and S. In this case the cone goes entirely through the cylinder, so that the intersection breaks up into two parts.

PROBLEM 28. *To find the intersection of two cones.*

See Fig. 104. The sectioning planes must contain the vertex of each cone. Hence the line ab, joining the vertices of the cones, is projected as a point, here shown in the third projection. Through this point the edge views of the sectioning planes are drawn, after which the procedure is similar to that of the preceding two problems.

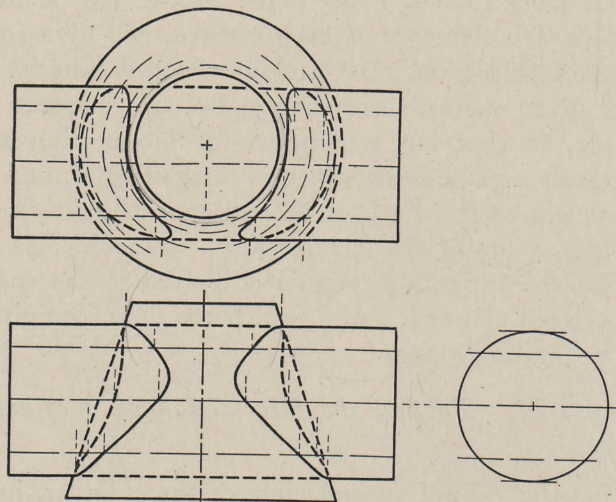


Fig. 105.

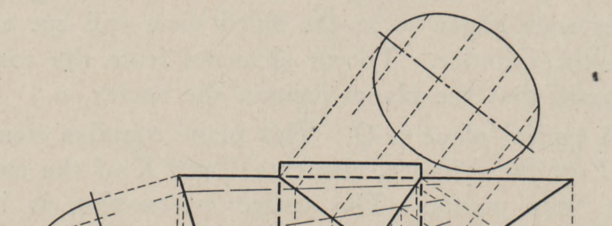


Fig. 106.

94. *Special Constructions.* In particular cases, instead of using sectioning planes which cut straight lines from both of the given surfaces, it may be more convenient to cut circles from one of them. An example is given in Fig. 105, in which horizontal planes seen edgewise in the elevation cut the cylinder in straight lines and the conical frustum in circles.

95. *Intersections Consisting of Plane Curves.* It is easy to see how two curved surfaces may intersect in a plane curve. Thus, a plane section of a hyperbolic paraboloid may be a parabola. Then, if we cut an equal parabola from a suitably chosen cone, we have the possibility of a cone intersecting a hyperbolic paraboloid in a parabola. Or, a cone may intersect a sphere in a circle, and so on.

In the case of cylinders and cones of the usual forms, the intersection will consist wholly of plane curves if the two surfaces are tangent to each other at two separate points. Thus, in Fig. 106, the given cylinder and cone are tangent to each other at the two points a and b. The plan of the intersection, found in the usual manner, will be found to consist of two straight lines, which must be interpreted as the edge views of two ellipses common to the two surfaces. Plane curves also result when one or both of the tangent points are at infinity. Thus, two equal cones of revolution, placed as shown in Fig. 107, will intersect each other in the two branches of a single hyperbola, and there will be no other intersection.

Cylinders and cones intersecting each other in plane curves are of frequent occurrence in intersecting arches and pipes, and in sheet metal work, such as funnels, elbows, and ship ventilators.

96. *The Intersection of a Plane with a Surface of Revolution.* Sectioning planes taken perpendicular to the axis cut the surface in a circle (or circles). But these circles will project as circles only if the axis of the surface projects

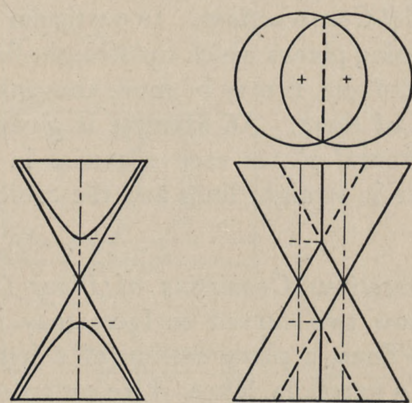


Fig. 107.

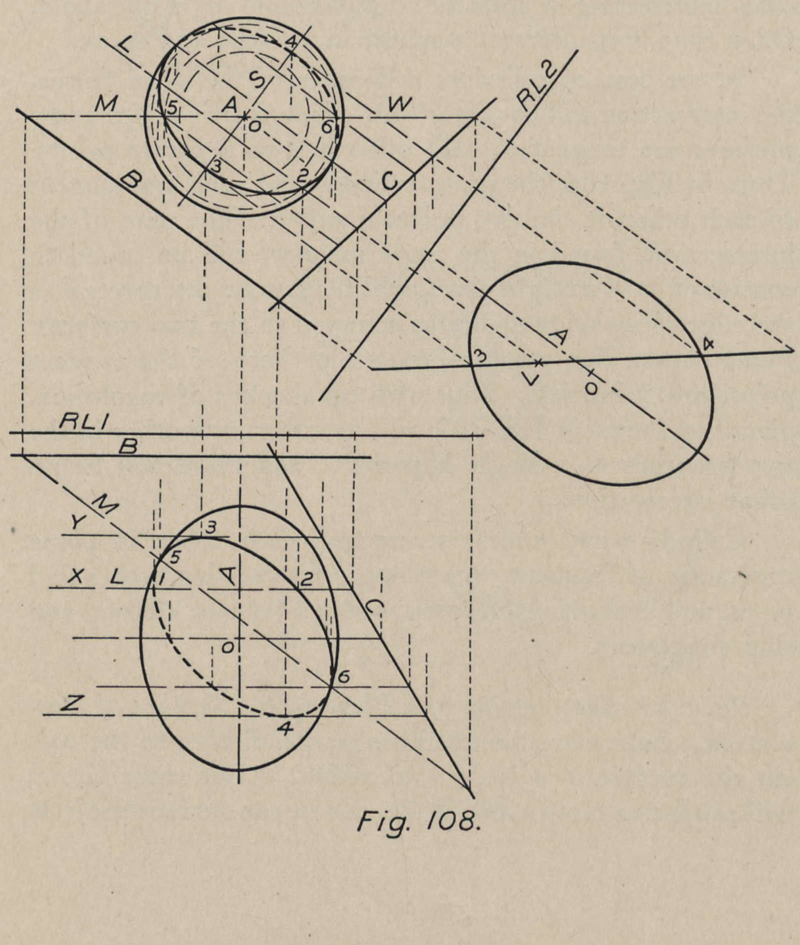


Fig. 108.

as a point. Hence in the following problem we shall assume the axis of the given surface to be perpendicular to one of the planes of projection, for if not so given, it can be so projected.

PROBLEM 29. *To find the intersection of a surface of revolution and a plane.*

See Fig. 108. The given surface is an ellipsoid with the axis vertical. The axis therefore projects as a point in the plan. The given plane is that of the intersecting lines B and C. A sectioning plane such as X, perpendicular to the axis, cuts the ellipsoid in a circle and the plane in a straight line, whose intersections in the plan determine two points. Other points, sufficient to determine the entire intersection, may be similarly obtained. However, certain critical points can be found to advantage by constructions which obtain them directly.

Points on the axis of symmetry. After a number of points have been found, it becomes evident in the plan view that the intersection is symmetrical about the line S, which is perpendicular to the line L. But the line L projects in true length in the plan, so that a view taken perpendicular to L gives an edge view of the given plane BC. If, therefore, such a view is made, as shown, and the outline of the surface drawn in this view, we have at once the intersections *3* and *4*. In this case, these points are the highest and lowest points of the curve.

Otherwise, if the points *3* and *4* are found by trial, it is necessary to locate the planes Y and Z so that the circle cut from the curved surface is tangent to the straight line cut from the plane.

Points on the outline of the elevation. Points *5* and *6*, on the outline of the elevation, separate in that view the visible from the invisible part of the intersection. These points may also be found by direct construction. In the plan, draw through the axis of the surface the horizontal

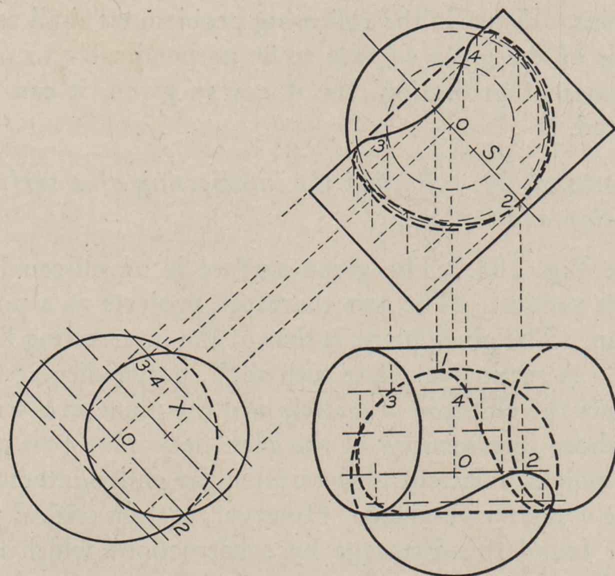


Fig. 109.

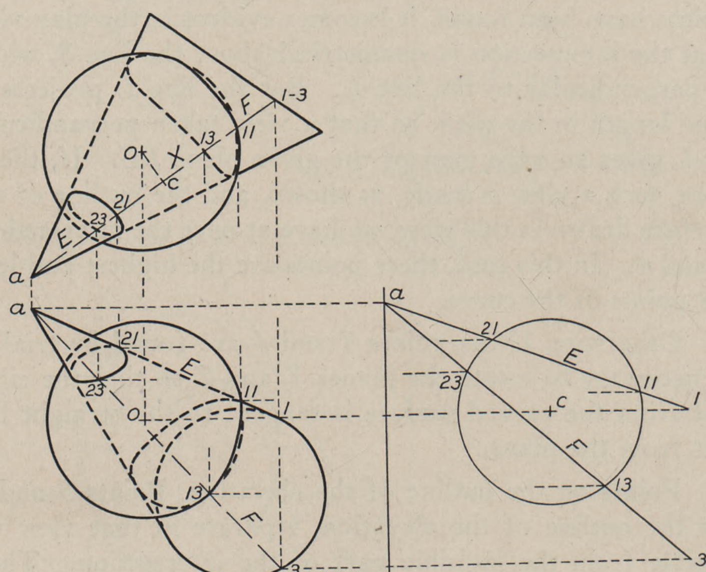


Fig. 110.

line W. Let this represent the edge view of a sectioning plane. This plane cuts from the surface the curve which appears as the outline of the elevation, and from the given plane BC the line M. Projecting to the elevation, we have at once the intersections *5* and *6*.

Note. This method can be utilized to find a plane section of any surface of revolution, whether single curved, double curved, or warped.

97. *The Intersection of a Sphere with Another Surface.* Any plane section of a sphere is a circle. Hence sectioning planes are chosen so as to cut simple figures from the second surface.

If the second surface is one of revolution, the sectioning planes would generally be taken so as to cut circles from that surface. But the sphere is itself a surface of revolution. This case, therefore, falls under the general problem of the intersection of two surfaces of revolution, and will not be discussed separately.

In the following two problems the second surface is single curved, and so placed that the solution can best be effected by the use of sectioning planes which cut that surface in straight lines.

PROBLEM 30. *To find the intersection of a sphere and a cylinder.*

See Fig. 109. Project the cylinder until its axis appears as a point, as shown in the third view in Fig. 109. In this view may be drawn the edge views of sectioning planes, each of which cuts the sphere in a circle and the cylinder in rectilinear elements. A typical plane is X, which gives the points *3* and *4* of the intersection. The points *1* and *2* on the axis of symmetry appear at once in the third projection.

PROBLEM 31. *To find the intersection of a sphere and a cone.*

See Fig. 110. Sectioning planes passed through the vertex and base of the cone will cut the cone in straight

lines and the sphere in circles. But such planes will not be parallel, and there is no single view which can be made which will show all the circles cut from the sphere simultaneously as circles. Therefore we must make a separate true size projection for each plane.

The treatment of a typical plane, X, is shown. This plane contains the elements E and F of the cone, and a circle of the sphere, whose center, c, is obtained by dropping from o, the center of the sphere, a perpendicular to plane X. The true lengths of the elements E and F are constructed as in the development of the cone (Prob. 18); the center c located, and the circle of proper size drawn. The four points thus determined are then carried back to the elevation and plan. The other points are found in a similar manner.

Note. This is the general solution. A simpler solution may be had if circles can readily be cut from the cone. For in this case the sectioning planes will be parallel, and a single view can be obtained in which all the circles appear as circles.

98. *The Intersection of Two Surfaces of Revolution.* If two surfaces of revolution have parallel axes, sectioning planes may be passed which will cut the surfaces simultaneously in circles. In particular, if one of the surfaces is a sphere, the planes are perpendicular to the second surface.

If two surfaces of revolution have a common axis, their intersections, if any exist, can consist only of circles lying in planes perpendicular to the common axis. In particular, since any diameter of a sphere may be taken as its axis, if the center of a sphere lies on the axis of any surface of revolution, the two surfaces can intersect only in circles.

If the axes of two surfaces of revolution intersect in a point o, a plane perpendicular to either axis will not, in

general, cut a circle from the other surface. But concentric spheres whose centers are at point *o* can intersect the given surfaces in circles, and there may be spheres which intersect both surfaces simultaneously.

If the axes of two surfaces of revolution do not lie in the same plane, there are, in general, no planes or spherical surfaces which can cut the given surfaces simultaneously in circles; and at most, only a very limited number.

From the above it follows that the intersection of two surfaces of revolution can be found by the use of sectioning planes if the axes of the surface are parallel, and by sectioning spheres if the axes intersect. This applies to all surfaces of revolution, single curved, double curved, or warped, although with single curved surfaces it may not give the simplest solution.

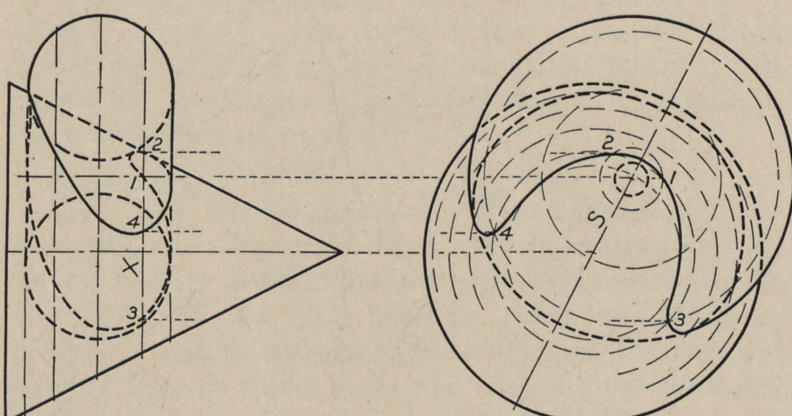


Fig. 111.

PROBLEM 32. *To find the intersection of two surfaces of revolution.*

Case I. The axes of the surfaces are parallel. See Fig. 111. The axes should be given, or projected, so as to appear as points in one of the views. In the figure this view is the front elevation. Sectioning planes perpendicu-

lar to both axes can be drawn in edge view in the side elevation. A typical plane is X, which cuts a circle from the cone and two circles from the torus. These circles determine four points of the required intersection.

Critical points. The plane S, which contains the axes of the two surfaces, is evidently a plane of symmetry. Points on this plane can be found, if desired, by projecting

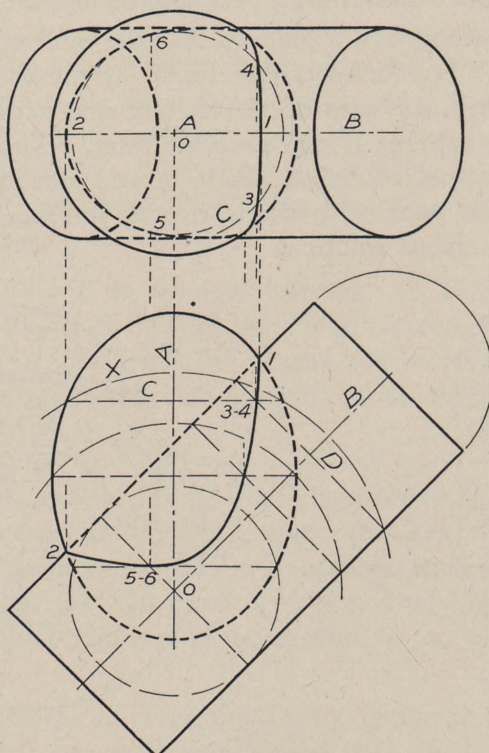


Fig. 112.

a normal view of the plane. This construction is not shown in Fig. 111, and usually is not necessary, the intersection being sufficiently determined otherwise.

Case II. The axes of the surfaces intersect. See Fig. 112. The given surfaces are an ellipsoid and cylinder of

revolution, whose axes, *A* and *B*, intersect at the point *o*. One view of the axes should be a point, here accomplished by taking the axis of the ellipsoid vertical. Point *o* is taken as the center of sectioning spheres, which are drawn only in the elevation. A typical sphere is *X*. This intersects the ellipsoid in a circle *C*, and the cylinder in a circle *D*, both of which project in edge view in the elevation. Their intersection in the elevation represents two points, *3* and *4*. These points are found in the plan by drawing only the circle *C*, since *D* would project as an ellipse.

Case III. The axes of the surfaces are neither intersecting nor parallel. In this case, the method of finding the intersection will vary, according to the nature of the surfaces. If both surfaces are single curved, that is, cylinders or cones, it is not necessary to treat them as surfaces of revolution, and solutions covering the various cases have already been given. But if one or both of the surfaces is double curved, the situation falls under the general problem of the intersection of any two surfaces. This will be taken up as the next problem.

99. *The Intersection of Any Two Curved Surfaces. General Case.* Reviewing the preceding constructions, we see that in every case we have been able to cut simultaneously from each of the given surfaces either straight lines or circles. But, to say nothing of surfaces on which neither straight lines or circles exist, and confining our attention solely to ruled surfaces and surfaces of revolution, there are numerous cases where simultaneous sections of straight lines and circles are not possible, or else too few to determine the intersection of the surfaces. In such cases we must accept other sections, such as ellipses or hyperbolas, on one of the given surfaces. Since these curves must be found by points, necessitating an additional construction, the sectioning surfaces should if possible cut straight lines or circles from the other given surface.

Each case must be worked out on its merits, and the following example should be regarded merely as typical.

PROBLEM 33. *To find the intersection of two curved surfaces, when none of the preceding methods apply.*

See Fig. 113. The surfaces chosen to illustrate this problem are a cone of revolution with its axis vertical, and an ellipsoid of revolution with its axis horizontal.

We first seek the planes, if any, which cut both surfaces in simple sections. Such planes are X, Y, and Z.

Plane X, through the center of the ellipsoid perpendicular to the axis of the cone, cuts the cone in a circle, and the ellipsoid in the ellipse which is the boundary of its plan. The intersections in the plan determine the points *1* and *2*.

Plane Y through the axis of the cone cuts the cone in the two elements which are the boundary of its front elevation, and the ellipsoid in an ellipse. The intersections in the front elevation determine points *3* and *4*.

Plane Z through the axis of the cone cuts the cone in the two elements which are the boundary of its side elevation, and the ellipsoid in a circle. The intersections in the side elevation determine points *5* and *6*.

We now seek other planes to complete the intersection. Planes can be drawn edgewise through the vertex of the cone in any one of the views. Such planes will cut straight lines from the cone; but, except for the plane Z already used, the sections of the ellipsoid are ellipses, difficult to obtain. Planes parallel to X or to Y will also cut the ellipsoid in ellipses. But planes parallel to Z will cut the ellipsoid in circles. Such planes will cut the cone in hyperbolas, but these curves are more easily constructed than elliptical sections of the ellipsoid. So that in this case planes parallel to Z are the best sectioning planes for completing the intersection.

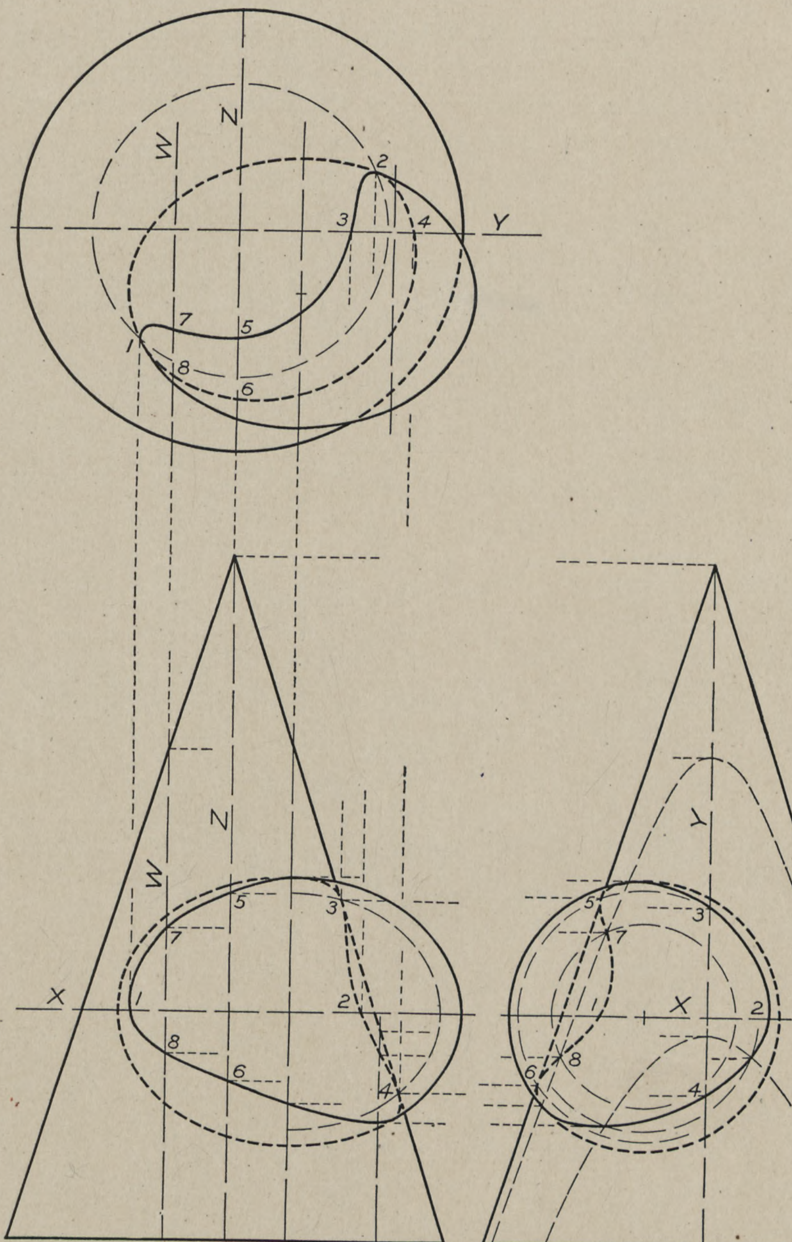


Fig. 113.

A typical plane is W. Points 7 and 8 in this plane are determined by the intersection of the circle and hyperbola shown in the side elevation. For the sake of clearness the construction of the hyperbola is not shown.

Chapter 8

Methods of Revolution

100. *The Construction Cone.* See Fig. 114. Let *abc* be a right triangle, the right angle being at *c*. If the triangle is revolved about one of its perpendicular sides, as *ac*, the result is a cone of revolution. The hypothenuse, *ab*, generates the conical surface, and the second perpendicular side, *bc*, generates the base, a circle. This cone possesses several useful properties.

1. Every element of the conical surface is of constant length; that is, any element *ad* is the same length as the hypothenuse *ab*.

2. Every element makes a constant angle with the axis; that is, any element *ad* makes with the axis *ac* the angle $\phi = bac$.

3. Every element makes a constant angle with the plane of the base; that is, any element *ad* makes with the plane of the base the angle $\theta = abc$.

4. The circle of the base lies on the surface of the sphere whose center is *a* and radius *ab*.

In order to be useful in the solution of problems, the cone must be in one of the simple positions shown in Fig. 114. That is, the base must project either as a circle or a straight line. The first results when the plane of projection is perpendicular to the axis of the cone; the second, when the plane of projection is parallel to the axis. So that, when this cone is used, additional views may be necessary.

101. *Notation.* The constructions of the present chapter can be explained to better advantage if we revert,

in a measure, to the notation of Chapter 1. We saw there that a plan is a view made on a horizontal plane, *H*, and an elevation a view on a vertical plane, *V*, at right angles to *H*. Data being given or assumed, the description of the solution will be made by using, instead of plan and elevation, the terms *H*-projection and *V*-projection. If then, in the data and solution, the letters *H* and *V* be reversed throughout, the description is equally true. Or, the letters *H* and *V* can be replaced by any two letters representing any two planes of projection at right angles to each other. Thus a single description can be adapted to any position of the given data.

In order to study the effect of data in some other position than that shown in the given figures, it may be of assistance to turn the figures upside down, and read them in this position, calling the view then at the top of the sheet the plan. Lines previously perpendicular to *H* will now be found perpendicular to *V*, and vice versa. Or, the given views may be turned sideways, and read as two elevations.

To aid in generalizing the solutions, the index letters *h* and *v* will not be used in the accompanying figures. This will render them less confusing when read from positions other than the bottom of the sheet, as suggested above. Instead, for purpose of reference, some of the views will be marked with primes and seconds. As this marking will be somewhat arbitrary, it will be necessary to interpret it in each figure as the occasion demands.

102. *The True Length of a Line.* One of the simplest applications of the construction cone, and one requiring no additional views, is in finding the true length of a straight line.

PROBLEM 34. *To find the true length of a straight line.*

See Fig. 115. Let *ad* be the line, projected as $a'd'$ and $a''d''$. Through one end of the line, as *a*, take an axis perpendicular to *H*. Revolve the given line about the axis, thus forming a cone of revolution; the point *d* will describe the base. The entire cone need not be drawn, merely enough to show one of the boundary elements, *ae*, whose *V*-projection, $a''e''$, equals the true length of *ad*. (Compare with Fig. 114.)

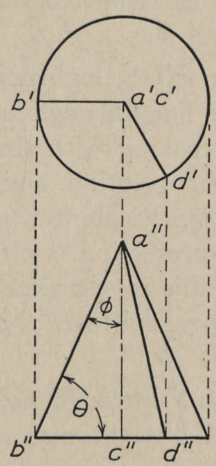


Fig. 114.

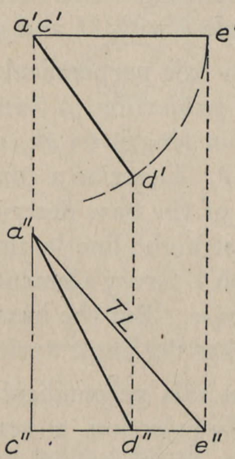


Fig. 115.

Reading the views in Fig. 115 as plan and elevation, from the triangle $a''c''e''$ we have at once the rule:

The true length of a line is the hypothenuse of a right triangle whose base is the length of its plan, and whose altitude is the difference in elevation between the two ends of the line.

This is the same rule previously derived by making a third view. See Lemma 3, Art. 50.

103. *Further Applications.* The principal use of the construction cone is in problems in which a line is required to make given angles with other lines or with planes. For,

as already pointed out, the elements of the cone make constant angles with its axis and its base.

PROBLEM 35. *To draw a line making given angles with the planes of projection.*

See Fig. 116. Let point a, projected as a', a'', be one end of the line, and the distance ak (not shown in the figure) be its true length. (If these data are not given, they must be assumed.) Let the line be required to make the (acute) angle θ with H and ϕ with V.

Take an axis perpendicular to H through point a, and draw its V-projection, a vertical line through a''. Draw $a''b''$ equal in length to ak, and making with the axis the angle $90° - \theta$. Describe a cone of revolution, and draw the projections of the base described by the point b'', namely, a horizontal straight line in the V-projection, a circle in the H-projection. Every element of this cone makes the angle θ with the base. But the base is parallel to H; hence every element makes the angle θ with H.

Take an axis perpendicular to V through point a, and draw its H-projection, a vertical line through a'. Draw $a'c'$ equal in length to ak, and making with the axis the angle $90° - \phi$. Describe a cone of revolution, and draw the projections of the base traced by the point c, namely, a horizontal straight line in the H-projection, a circle in the V-projection. Every element of this cone makes the angle ϕ with the base; that is, with V.

The bases of the two cones just drawn both lie on the surface of the sphere whose center is a and radius ak. So that, if the projections of the bases intersect, the intersections represent actual points, any one of which may be connected with point a as a solution of the problem. One such intersection is shown in the figure at point d, giving the line ad as a solution. If both nappes of both cones are drawn, there may be as many as eight points of intersection, occurring in pairs at the ends of four diameters of the sphere.

PROBLEM 36. *Through a given line, to pass a plane which shall make a given angle with one of the planes of projection.*

Let A be the given line, and let the plane be required to make the angle θ with the H-plane of projection.

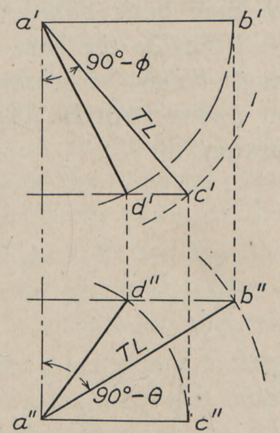


Fig. 116.

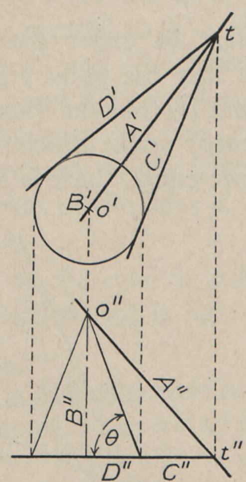


Fig. 117.

See Fig. 117. Assume any point o on A. Through o draw the line B perpendicular to H; its H-projection is a point, B'; its V-projection, B'', is a vertical line. With o as vertex, and B as axis, draw the projections of a cone of any convenient length, whose elements make with H the given angle θ. The base of this cone lies in a plane which shows in edge view in the V-projection, as a straight line perpendicular to B''. Produce this line to intersect A'' in t''. Project from t'' to t' in A'. From t' draw C' and D' tangent to the circular projection of the base. Since these tangents represent lines in the plane of the base, their V-projections, C'' and D'', coincide with the edge view of the base.

The plane of the lines A and C contains the vertex of the cone and a line tangent to the cone. The plane is,

therefore, tangent to the cone. Hence it makes with the plane of the base of the cone, that is, with *H*, the same angle, θ, as that made by the elements. Similarly for the lines *A* and *D*. There are thus two solutions; one, the plane of lines *A* and *C*, the other, the plane of lines *A* and *D*.

Note. In order that there may be a solution it is obvious that the point *t* must fall outside the base of the cone; that is, that the (acute) angle θ must be greater than the (acute) angle which the line makes with *H*. If these angles are equal, there is one solution.

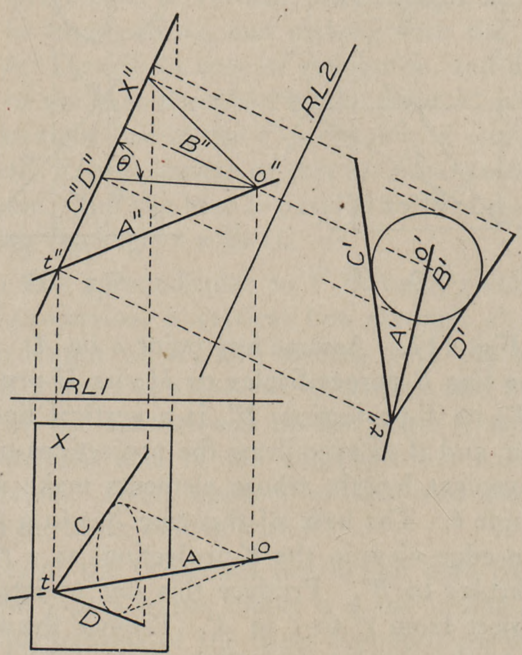


Fig. 118.

COROLLARY. *Through a given line to pass a plane which shall make a given angle with any given plane.*

See Fig. 118. Let *A* be the given line, and let the given plane, *X*, be perpendicular to *H*, so that it appears

edgewise in the H-projection. Find the point in which the line intersects the plane; the H-projection of this point is t'', where A'' *intersects* X''. Select any other point, o'', in A''. Draw B'' perpendicular to the plane X. Make B the axis of a cone whose base is in X, and whose elements make the given angle θ with X. To see the base of the cone as a circle, project on a plane parallel to X''. Project t'' *to* t', and draw the two tangents C' and D'. Since these represent lines in plane X, the H- and V-projections of C and D can now be found. As in the previous case, each of these lines determines a plane tangent to the cone. So that one of the required planes is that of the lines A and C, the other is the plane of A and D.

In Fig. 118, the V-projection of the cone is indicated. This is not essential to the construction, but is included as an aid in visualizing the result.

If the given plane X does not appear edgewise in either of the original views, it must be so placed by an additional projection. In this case four views in all will be needed to solve the problem.

PROBLEM 37. *To draw a line which shall make given angles with each of two given intersecting lines.*

Let A and B be the given lines, intersecting at point o, and let it be required to draw through o a line or lines which shall make the (acute) angle θ with A and the (acute) angle ϕ with B.

Basic Solution. See Fig. 119. Place the given lines A and B parallel to V, and in addition, let one of them, as B, be perpendicular to H, so that its H-projection is a point. In the V-projection, take the intersection o as a vertex, line A as axis, and draw a cone whose elements make the angle θ with A. Similarly, with o as vertex and B as axis, draw a cone whose elements make the angle ϕ with B. Find the lines (elements) in which these two cones intersect. That is, with o'' as center and any con-

venient radius, draw in the V-projection a sphere which will intersect each cone in a circle, whose intersections determine the required lines. In Fig. 119 the lines are D'' and E''. The H-projections, D' and E', are obtained by projecting the base of the cone B, since from the position of its axis this base projects as a circle. (Compare Prob. 32, Case II.)

Number of Solutions. In Fig. 119, it is evident that if the sum of the angles θ and ϕ is less than the angle between the lines A and B, the cones will not intersect each other, and there will be no solution. Also, in Fig. 119, if the second nappe of each cone be drawn, they will intersect each other in the lines D and E produced, giving no new results. So that in this case there are two solutions.

In Fig. 120, which shows but one view, the angle θ has been greatly increased, so that both nappes of cone A intersect a single nappe of cone B. This gives four solutions, the maximum number possible.

If the angles are such that the cones are tangent to each other, there will be three solutions, or one solution, according as the cones do, or do not, also intersect.

General Case. In the most general case, the lines A and B are given so that neither projection of either line shows in true length. See Fig. 121. To solve the problem by the preceding method, we must have a view in which one line appears as a point, and another in which both lines appear in true length.

Select one of the lines, as B, and project it, first in true length, then as a point, letting the line A come as it will. The result is the fourth view, marked with primes, in Fig. 121. Project again on a plane parallel to A', giving the fifth view, in which A'' and B'' are both true length. The fourth and fifth views correspond to the two views of Fig. 119. In them the problem is solved, and the results carried back to the original views. With the data as given,

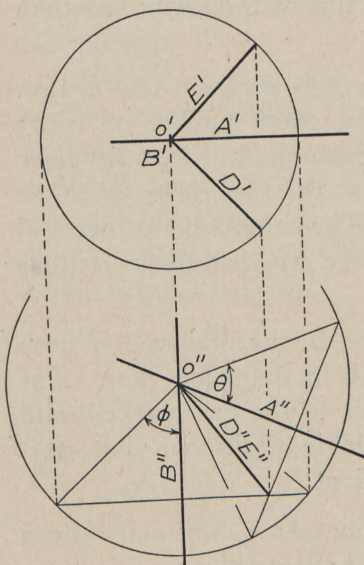


Fig. 119.

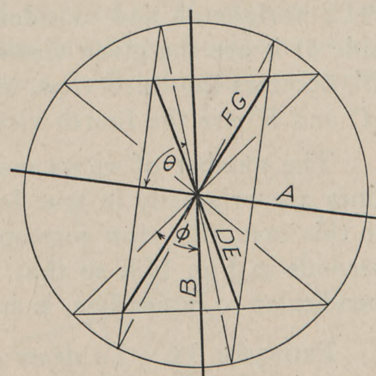


Fig. 120.

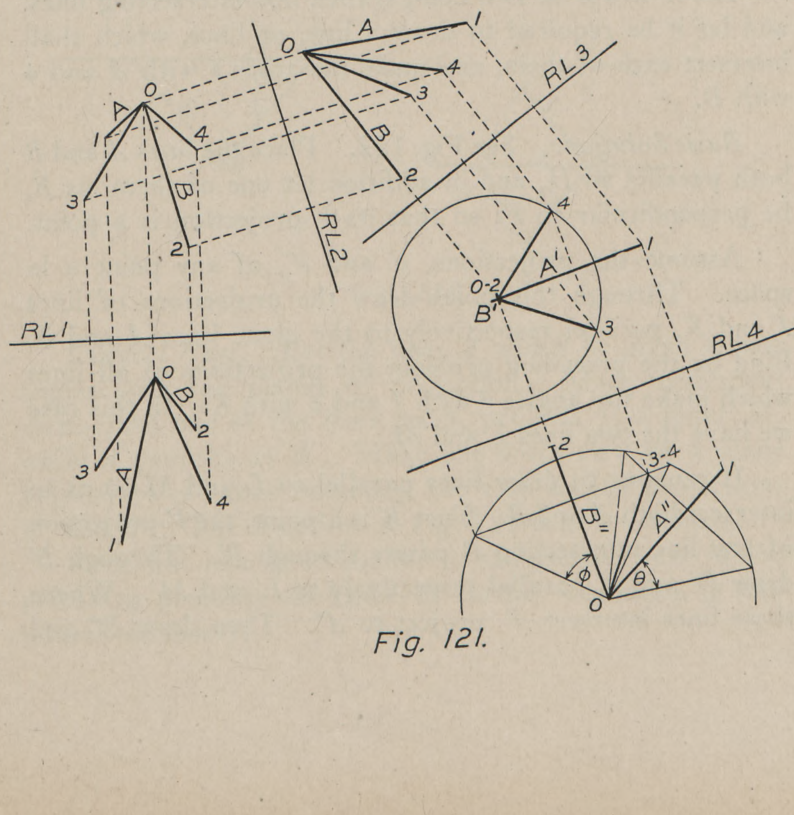


Fig. 121.

the problem cannot be solved by this method with less than five views.

But if one of the lines A or B is given in true length in one of the original views, the total number of views necessary can be reduced. For example, in Fig. 121, place $RL2$ horizontal, and consider that the two views on either side of it are the given views. Then the point projection, B', becomes the third view, and the true length projections A'' and B'' are the fourth view.

The number of views may be further reduced if both lines are originally in true length in the same view. For in this case, this view corresponds to the one marked with seconds in Fig. 119, so that a single additional view, perpendicular to either line, is sufficient.

PROBLEM 38. *To draw a line which shall make given angles with any two lines not in the same plane.*

Let A and B be two non-parallel, non-intersecting lines, and let it be required to draw a line, or lines, which shall intersect each of them, and make the angle θ with A and ϕ with B.

Basic Solution. See Fig. 122. Place the lines A and B both parallel to H, and in addition let one of them, as B, be perpendicular to V, so that its V-projection is a point.

Assume the projections, o' and o'', of any point o in space. Through this point draw the projections of lines J and K, parallel respectively to the given lines A and B. Find by the preceding problem the projections of all lines which make the angles θ with J and ϕ with K. In this case we have the two lines L and M.

It remains to draw lines parallel to L and M so as to intersect both A and B. Since B' is a point, the V-projection of any line intersecting B passes through B'. Through B' draw D' and E' parallel respectively to L' and M'. Where these lines intersect A', project to A''. Then draw D'' and

E'' parallel to L'' and M''. Lines D and E are solutions of the problem.

Number of Solutions. The number of solutions will be zero, two, or four. If the two cones are tangent in the auxiliary construction, the common element is parallel to H, so that a line through B, parallel to H, cannot intersect A. Hence the case of the preceding problem which has one solution gives none here, while the case of three solutions gives but two.

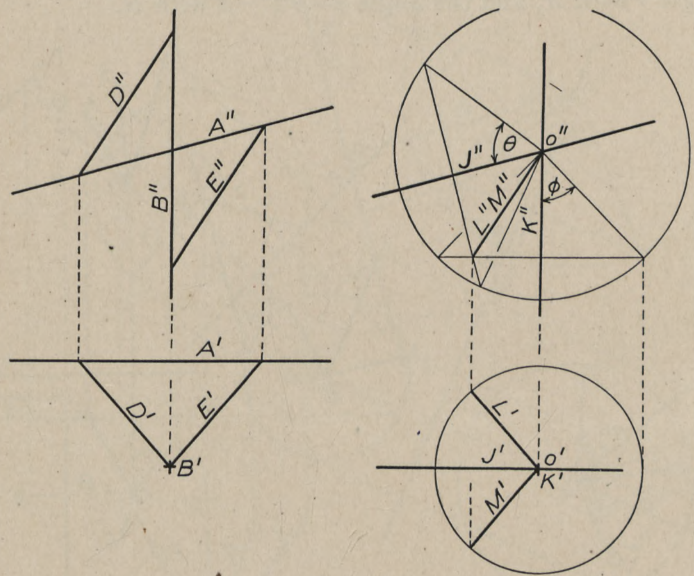


Fig. 122.

General Case. Let the lines A and B be given at any angles with the planes of projection. Make additional views until one of the lines, as B, projects as a point, B', while A projects as A'. Make one more view on a plane parallel to A', thus obtaining the true length projections A'' and B''. The situation of Fig. 122 is now attained.

Since this case is so similar to the general case of the preceding problem, no figure or further discussion is considered necessary.

PROBLEM 39. *To draw a line which shall make a given angle with a given line and a given angle with a given plane.*

Let A be the given line, and X the given plane. Let it be required to find a line, or lines, which shall make the angle θ with A, and β with X.

Basic Solution. (No figure.) Choose any point, o, on A. Through o draw a line, B, perpendicular to the plane X. Find, by Problem 37, the lines which make the given angle θ with A, and the angle $\phi = 90° - \beta$ with B.

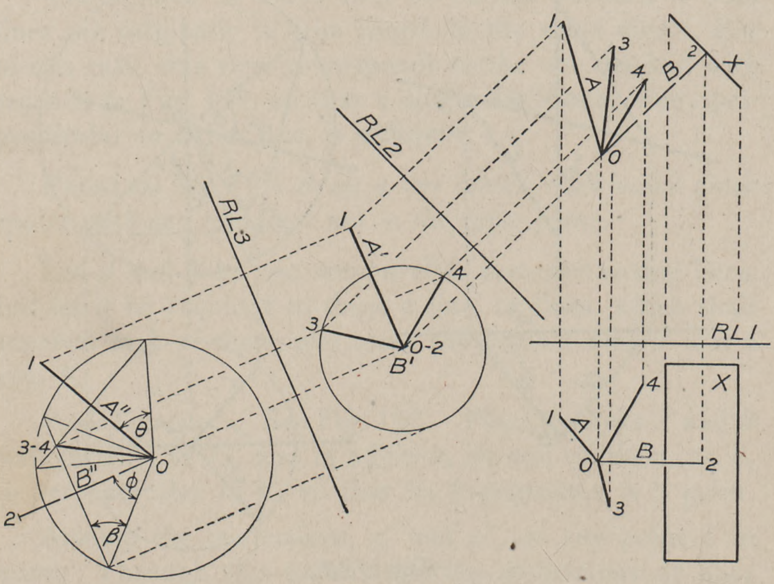


Fig. 123.

General Case. See Fig. 123. Let the line A and the plane X be given as shown. From any point, o, on A, draw the projections of a line B perpendicular to X. Project B as a point, B', and in this view let A project as A'. Project again on a plane parallel to A', obtaining the true length projections A'' and B''. The two views just obtained show the same situation as in Fig. 119. Solve in these two views; then project back to the original views.

In Fig. 123, the plane is given edgewise in one of the original views, so that four views in all are sufficient. If the plane is not so given, it should be so projected in the first additional view. In this case, five views in all will be needed.

COROLLARY. *To draw a line making a given angle with a given line and a given angle with one of the planes of projection.*

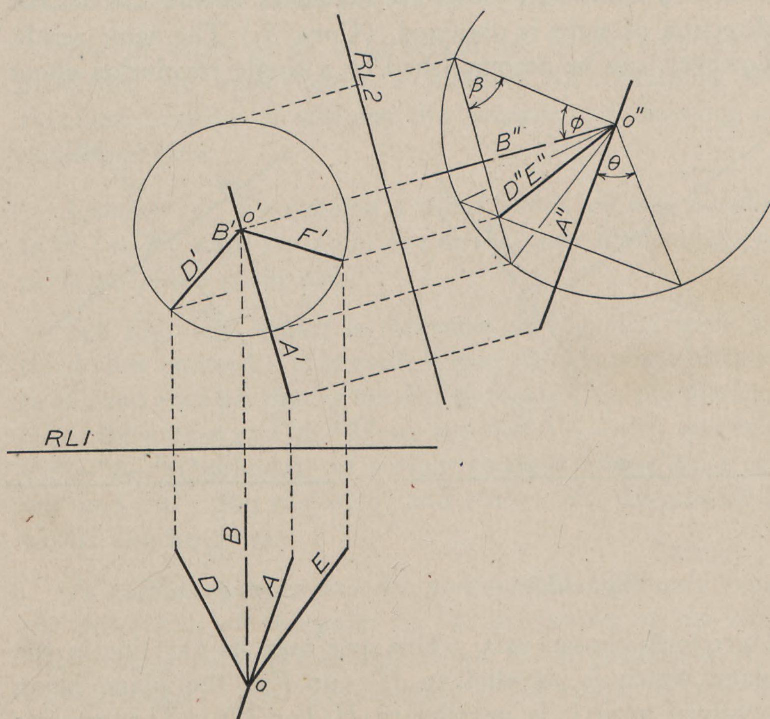


Fig. 124.

Let A be the given line, and let it be required to draw a line or lines making the angle θ with A and the angle β with the H-plane of projection.

See Fig. 124. Assume any point o on A. Through o draw a line perpendicular to H; its H-projection is a point,

B'. Project on a plane parallel to the H-projection of A, giving the true length projections A'' and B''. The H-projection and the third view give the situation of Fig. 119. Solve as in the general case.

104. *Revolution about an Axis Parallel to H or V.* Let a plane be oblique to both H and V, and let it be required to view the plane normally, that is, in a direction at right angles to the plane. By the method of successive projection, two additional views are necessary before the desired direction of sight is obtained. (Cons. 5.) The same result, however, can be accomplished by a single revolution about

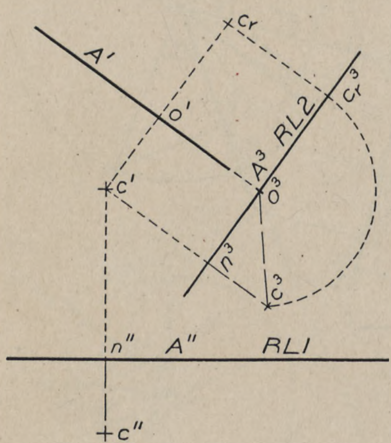


Fig. 125.

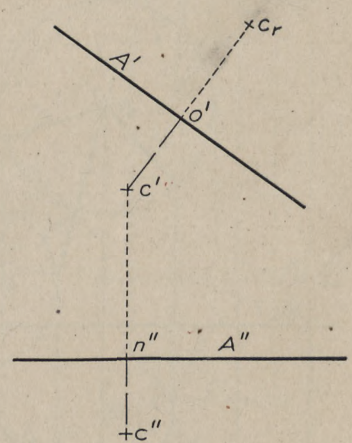


Fig. 126.

a properly chosen axis. This axis may be any line in the plane which is parallel to H (or V), the plane being revolved until it is parallel to H (or V). This process furnishes a more rapid method for solving certain problems which depend on the normal view of a plane.

105. *Revolution of a Point.* The fundamental process in this method is the revolution of a single point about the chosen axis. Let the line A, Fig. 125, be parallel to H but oblique to V. The plane parallel to H which contains

the line A will project edgewise as A''. Let c be any point not in this plane, and let it be required to revolve c about A as an axis until c lies in this plane. The path of the moving point is the arc of a circle lying in a plane perpendicular to A. In Fig. 125 two views of this path are drawn. The H-projection is a straight line, from c' perpendicular to A'. A third view, taken perpendicular to A', shows the path in true size, and enables us to locate c_r in the H-projection. This is the required revolved position.

But in order to be of practical use in the solution of problems, we must devise a method for obtaining the revolved position c_r without the necessity of drawing an additional view.

LEMMA 5. *To revolve a point about an axis parallel to H (or V) until the point lies in the same horizontal (or vertical) plane as the axis.*

See Fig. 126, which is the same as Fig. 125 with the third view omitted. It is evident that $c'o'c_r$ is perpendicular to A', and that all that is needed to locate c_r is the distance $o'c_r$. Reverting to Fig. 125 we see that $o'c_r = o^3c^3$, and that o^3c^3 is the hypothenuse of a right triangle whose sides are n^3o^3 and c^3n^3. *But* $n^3o^3 = c'o'$, and $c^3n^3 = c''n''$. Expressed in words, this becomes:

In each projection find the perpendicular distance from the projection of the point to the projection of the axis. Using these distances as sides, find the hypothenuse of a right triangle. This hypothenuse is the radius of the arc in which the point revolves.

106. *Normal View of a Plane.* The normal view of a plane or plane figure may be obtained by revolving a sufficient number of points.

PROBLEM 40. *To find a normal view of a plane by revolution.*

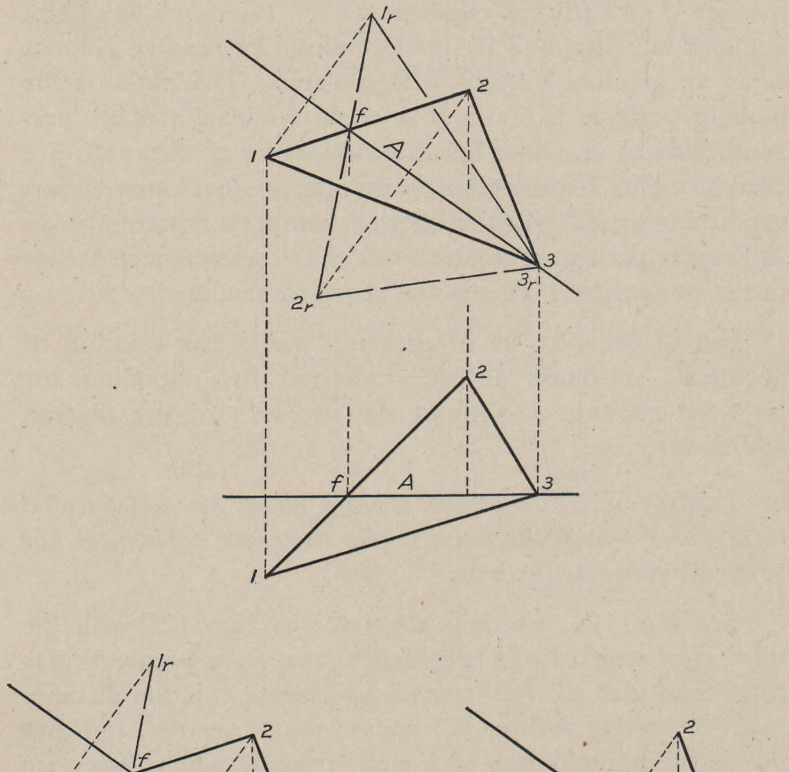


Fig. 127.

See Fig. 127. Assume an axis by drawing in the plane a line A, parallel to H (or V). That is, draw a line in the plane, one of whose projections is a true length view of the line (Lemma 2). Using this line as axis, revolve the various points of the plane by the preceding Lemma.

It is often convenient to take the axis through one of the given points, since the position of any point on the axis is unchanged by revolution. Thus, in the figure, the axis passes through the point *3*, which thus becomes its own revolved position. But if, as shown in the figure, the axis is so taken that parts of the plane lie on either side of the axis, care should be observed to preserve the same relative position of the parts after revolution.

107. *Solution of Problems.* The normal view of a plane obtained by revolution gives at once an alternate solution of the following problems of Art. 45, which by the method of successive projections require two additional views.

Problem 10 (repeated). To find the true size of a plane figure.

This has been already accomplished in Fig. 127, the result being the true size of the triangle *1·2·3*.

Problem 11 (repeated). To find the true angle between two intersecting lines.

See Fig. 127. Let the required angle be the angle between the lines *1·2* and *2·3*. Draw the axis A, and revolve the point *2*. This is the only point which need be revolved, the others being on the axis. The true angle is shown at 2_r.

Let the required angle be the angle between the lines *2·3* and A; that is, let one side of the angle be a line which projects in true length in one of the views. Then the true length view may be taken as the axis, and the second line revolved by revolving one of its points. In Fig. 127 the true angle is 2_r3_rA.

Chapter 9

Traces

108. *Traces.* A trace of a line or of a surface is its intersection with a plane of reference. Thus, the trace of a line is a point. The trace of a surface is a line, straight or curved, according to the form of the surface. In particular, the trace of a plane is a straight line.

109. *Determination of a Plane by Traces.* See Fig. 128. Let us establish a horizontal reference plane, H, and a vertical reference plane, V, by drawing their line of intersection, HV. This is the same as the reference line, RL or $RL1$, previously used. Let a plane, Q, oblique to both H and V, intersect H in the line HQ, and intersect V in the line VQ. Since no two of the planes H, V, and Q are parallel, the three planes intersect in a point. Hence HQ and VQ intersect each other in a point on HV. That is, HQ and VQ are two intersecting lines in the plane Q. These lines may therefore be used to determine, or locate, the plane Q. In accordance with the above definition, they are called the traces of the plane Q.

The trace of a plane must not be confused with the edge view of a plane. The edge view is a line which is a complete projection of the plane. The trace of a plane is, in general, merely a line of intersection, and can not be an edge view unless the given plane is perpendicular to the plane of reference.

110. *Special Positions of the Plane.* The plane R, shown in Fig. 129, is perpendicular to H and oblique to V. The trace on H, HR, is also an edge view of R. The trace on V, VR, is perpendicular to HV, but is not an edge view of the plane.

The plane *S*, Fig. 130, is oblique to both *H* and *V*, but is parallel to their intersection *HV*. The traces, *HS* and *VS*, are both parallel to *HV*. Neither trace is an edge view of *S*. The plane *S* slopes downward and forward. The distances of *HS* and *VS* from *HV* will vary according to the steepness of the plane *S*; but so long as *HS, HV,* and *VS* continue to maintain the same relative positions, the plane *S* will slope downward and forward.

The plane *T*, Fig. 131, is also parallel to *HV*, but slopes downward and backward. It is impossible to represent a plane of this slope by traces on *H* and *V* unless one of the reference planes is produced. Here *V* is produced, and the trace *VT* is found on the upper part of *V*.

The plane *X*, Fig. 132, is parallel to *V*. It is therefore perpendicular to *H*, and its single trace, *HX*, is also its edge view. Similarly, the plane *Y*, Fig. 133, is parallel to *H*. It is therefore perpendicular to *V*, and its single trace, *VY*, is also an edge view.

Other special positions of the plane, not shown in the figures, occur when the plane is perpendicular to the reference line *HV*, and when the plane contains the line *HV*. In the former case the plane is a profile plane (Art. 16). In the latter case the method of location by means of traces fails, so far as the given reference planes are concerned.

111. *The Method of Traces.* In the earliest text books on descriptive geometry, all planes were handled by their traces. The only edge views used were those of planes perpendicular to the planes of reference. Normal views of oblique planes, when needed, were obtained by revolution (Arts. 106, 121). In distinction to the method of obtaining edge and normal views of planes by successive projection, the former may be called the method of traces.

But if planes are handled exclusively by their traces, there are, of necessity, numerous special cases and auxiliary constructions. Further, it is not always possible to confine

the constructions within fixed limits. On this account, visualization of the construction may become difficult. *Nevertheless, for some problems, the neatest and shortest solutions are found by the method of traces. There are problems readily solvable by this method in the original plan and elevation which by the method of successive projection require two additional views.*

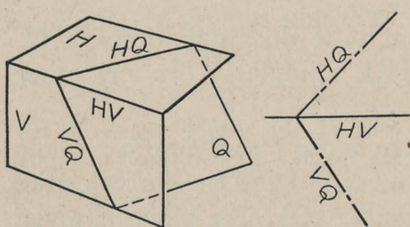


Fig. 128.

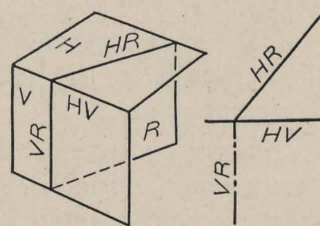


Fig. 129.

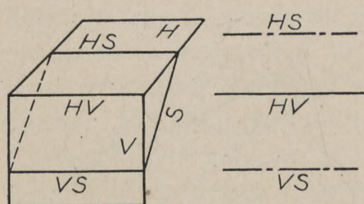


Fig. 130.

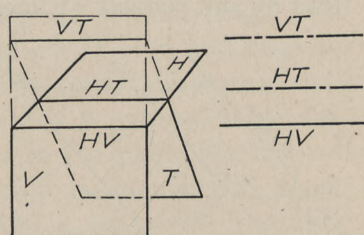


Fig. 131.

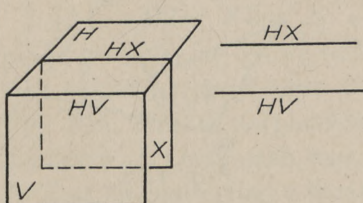


Fig. 132.

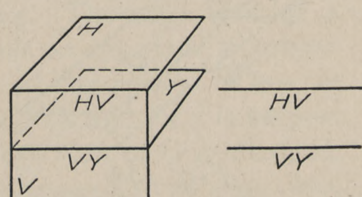


Fig. 133.

In consideration of the above, a full and comprehensive treatment of the method of traces will not be given here. The method will be developed only in so far as it may serve as a useful adjunct to the method of successive projection.

112. *Relation between Traces of Lines and Planes. Principle. If a line lies in a plane, the traces of the line lie in the corresponding traces of the plane.*

This is merely another way of saying the following: Let Q be any plane, and H a plane of reference. Let Q intersect H in HQ. Let A be any line in Q. Then if A intersects H, it can do so only in the line of intersection, HQ, of H and Q.

We may further note that if A does not intersect H, it is parallel to H, and to HQ. Similarly for any other reference plane.

If the projections of two or more lines lying in a plane are given, the principle may be used to find the traces of the plane on any planes of reference. Or, if a plane is given by its traces, the principle will enable us to find the projections of any number of lines in the plane.

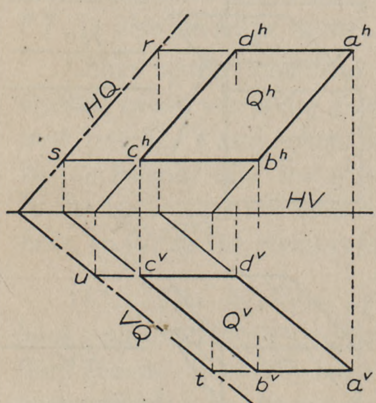


Fig. 134.

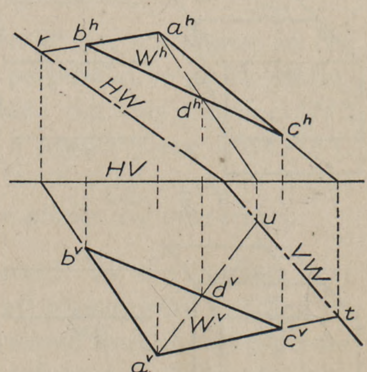


Fig. 135.

113. *To Find the Traces of a Plane.* Let us speak of the intersection of a line or plane with H as its H-trace; its intersection with V as its V-trace. The H-trace of a plane is known as soon as two of its points are known. In accordance with the foregoing principle, these points may be the traces of any two lines in the plane.

See Fig. 134. Let the plane *Q* be given as the parallelogram *abcd*. Let the planes of reference, *H* and *V*, be established by the line *HV*. In the elevation, this line is interpreted as the edge view of *H*. Produce the elevations a^vd^v and b^vc^v to *HV*, and project these intersections to the plan. This locates the points *r* and *s*, which must lie in *H*, and are consequently *H*-traces of two lines in the plane *Q*. Hence the *H*-trace, *HQ*, of the plane passes through *r* and *s*. Similarly, in the plan, *HV* is interpreted as the edge view of *V*. Produce the plan views a^hb^h and d^hc^h to meet *HV*, and project to *t* and *u*. These points are *V*-traces of two lines in plane *Q*, so that the *V*-traces, *VQ*, passes through them. The traces *HQ* and *VQ* must intersect on *HV* (Art. 109).

The lines which are produced to meet *H* and *V* are not necessarily those which are given to locate the plane. Thus, in Fig. 135, the plane *W* is given as the triangle *abc*. Edge *ab* may be produced to intersect *H* at *r*, and edge *ac* produced to intersect *V* at *t*. The other points resulting from producing the edges of *abc* fall at unreasonable distances. We may therefore draw in the plane the line *ad* (Art. 34), which can be produced to intersect *V* at *u*. Then *VW* is drawn through *t* and *u*, and *HW* is drawn through *r* and the point in which *VW* intersects *HV*.

114. *Edge View of a Plane.* The edge view of a plane given by its traces will appear on any plane of projection taken perpendicular to one of the traces. Thus, in Fig. 136, *RL2* is perpendicular to *VQ*, and *RL3* is perpendicular to *HQ*; in each case, an edge view of *Q* is obtained. At first, in converting a location by traces into an edge view, it is well to draw a triangle in the plane by selecting three points, as shown by *abc* in the figure. But as soon as it is recognized that the trace projects as a point, it is necessary to use but one other suitably chosen point in the plane. For example, point *c* when *RL2* is used, point *b* when *RL3* is used.

In Figs. 137 and 138 the given plane is parallel to HV. The edge view shows in the side elevation (profile projection). Note that in Fig. 138 it is impossible to confine the construction within the usual limits, although a point such as e, situated in the given plane, would have its plan, front, and side elevations placed as usual.

115. *Intersection of Planes.* The line of intersection of two planes given by their traces may be found from the principle of Art. 112, namely, that *if a line lies in a plane the traces of the line lie in the corresponding traces of the plane.* For, if a line lies in each of two planes, its trace on any plane of reference must lie in the trace of each of the given planes, and consequently must be their point of intersection.

A general case is shown in Fig. 139. The intersection of the H-traces HQ and HR locates t^h, the plan of a point common to the two planes. *This point lies in H, so that* its elevation, t^v, must be on HV. The intersection of the V-traces, VQ and VR, locates s^v, the elevation of a point which lies in the two planes and in V. The plan, s^h, lies in HV. The line of intersection, A, can now be drawn, since two points in each view have been found.

Other general cases are given in Figs. 140 and 141. In each case the lettering and description are the same as in Fig. 139, but some of the construction lies on that portion of V which is above the reference line HV.

In Fig. 142 the given plane Y is parallel to H. The line of intersection, B, of Y and Q is therefore parallel to H. But since B lies in Q, and is parallel to H, it must be parallel to the H-trace, HQ (Art. 112). One point, then, will locate the line B. This point is s, the V-trace of the line. The plan, B^h, is drawn through s^h parallel to HQ. The elevation, B^v, coincides with VY, since this trace is also an edge view of the plane Y.

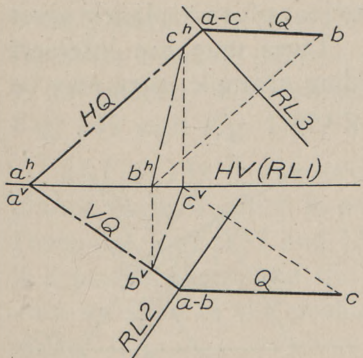
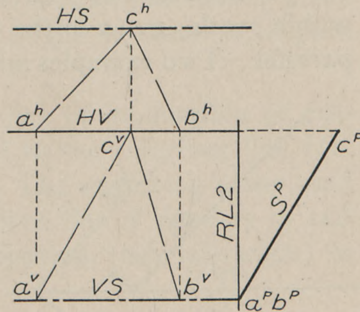


Fig. 136.



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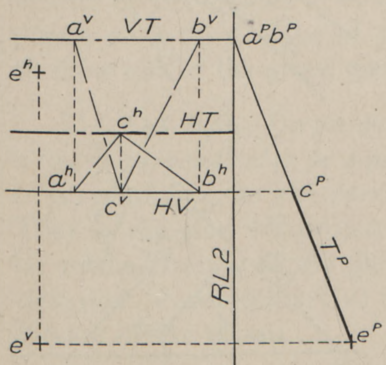

Fig. 138.

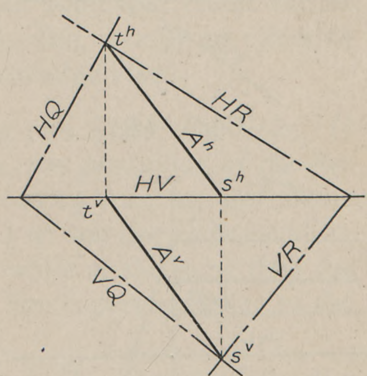

Fig. 139.

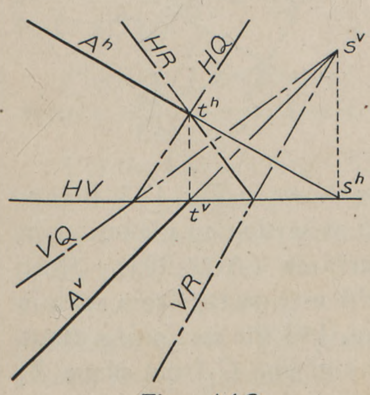

Fig. 140.

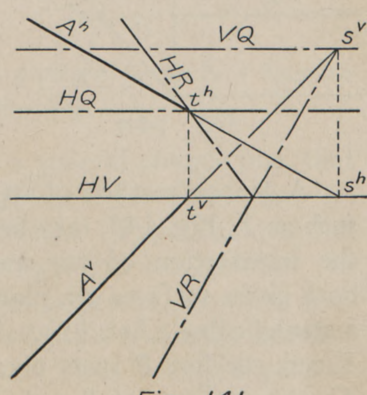

Fig. 141.

The problem of the intersection of two planes gives rise to numerous special cases. Lines may not intersect within reach, or, as in the preceding example, they may be parallel. Two examples will be given.

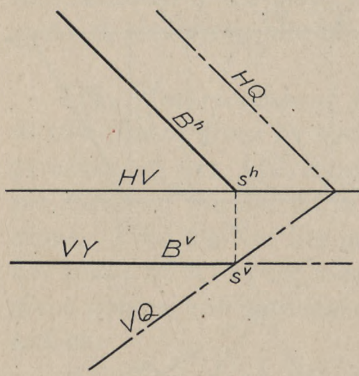

Fig. 142.

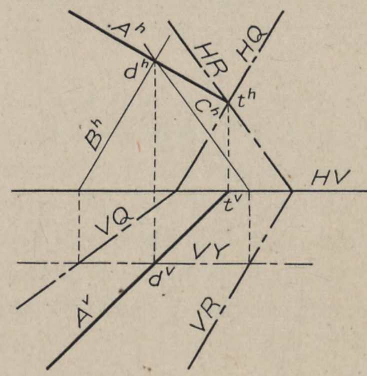

Fig. 143.

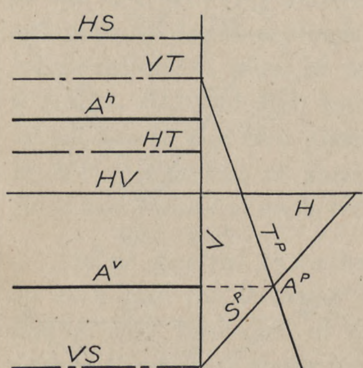

Fig. 144.

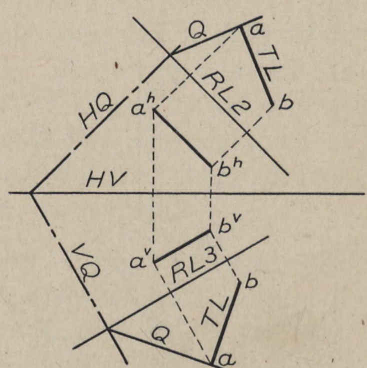

Fig. 145.

A general method of attack is shown in Fig. 143. Planes such as Y, Fig. 142, may be used as sectioning planes, as in the intersection of any two surfaces (Art. 90). Since both given surfaces are planes, the section cut from each is a straight line (Art. 91). In Fig. 143 the sectioning plane Y cuts the line B from plane Q and line C from plane R. The intersection of B and C is point d, a point common to

both Q and R. In this case the intersection of HQ and HR furnishes a second point, t, so that a second sectioning plane is not needed. Otherwise, one might be used. Compare Fig. 143 with Fig. 140.

In Fig. 144 is obtained the line of intersection of two planes, each of which is parallel to the reference line HV (compare Figs. 137 and 138). The sectioning plane used is a profile plane, and the required line A appears in end view as soon as the profile projection (side elevation) is made.

Another method of finding points in the line of intersection of two planes is to obtain additional traces of the planes on a third plane of projection, usually so taken as to show one of the given planes in edge view.

116. *A Plane Perpendicular to a Line. Principle. If a line is perpendicular to a plane, any projection of the line is perpendicular to the corresponding trace of the plane.* That is, the plan (H-projection) of the line is perpendicular to the H-trace of the plane, while at the same time the elevation (V-projection) of the line is perpendicular to the V-trace of the plane.

To prove this principle, let us again solve the following problem of Art. 41, this time locating the required plane by its traces.

Problem 1 (repeated). At a given point in a line, to draw a plane perpendicular to the line.

Let the line ab, Fig. 145, be a general oblique line, that is, one not projected in true length in either view. Let the plane Q be required to contain point a. Take $RL2$ parallel to the plan, a^hb^h, of the line, giving a true length view. In this view the plane Q will appear edgewise, as a line perpendicular to ab. The plane Q will then obviously intersect H in a line perpendicular to $RL2$. That is, HQ is perpendicular to $RL2$, hence to a^hb^h.

Also, take *RL3* parallel to the elevation, a^vb^v, of the line, obtaining again a true length view of *ab*. In this view the plane *Q* again appears edgewise, as a line perpendicular to *ab*, and obviously intersects *V* in a line perpendicular to *RL3*. That is, *VQ* is perpendicular to a^vb^v. Since *HQ* and *VQ* are the traces of the same plane, they must intersect on *HV*. Thus the principle is established.

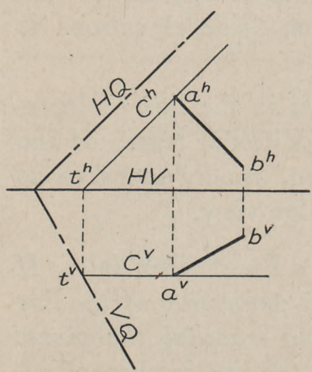

Fig. 146.

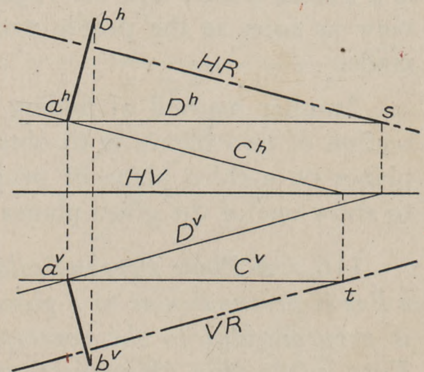

Fig. 147.

But once established, a much shorter construction can be used. In Fig. 146 the line *C*, which is in true length in the plan, is drawn so as to be at right angles to *ab* (Art. 42; C^h is perpendicular to a^hb^h). Then *C* is a line in the required plane *Q*. Find the *V*-trace, t^v, of *C*. Draw *VQ* through t^v, perpendicular to a^vb^v. If *VQ* intersects *HV* within reach, through this point draw *HQ* perpendicular to a^hb^h.

If the traces of the required plane do not intersect *HV* within reasonable limits, the construction of Fig. 147 may be used. Two lines, each perpendicular to *ab*, may be drawn; line *C* parallel to *H*, line *D* parallel to *V*. These lines completely determine the plane.

COROLLARY 1. *Through any point in space, to pass a plane perpendicular to a given line.*

Through the given point, draw a line parallel to the given line. Then pass the plane as in Fig. 146 or 147.

COROLLARY 2. *Through any point in space, to draw a line perpendicular to a given plane.*

Draw the projections of the line perpendicular to the corresponding traces of the plane. Note that this is a direct application of the basic principle.

117. *The Shortest Distance from a Point to a Plane.* The shortest distance from a point to a plane is measured in a line perpendicular to the plane. But since the traces of a plane are not, in general, edge views, the actual distance does not appear until a third view is made.

Problem 8 (repeated). To find the shortest distance from a point to a plane.

(No figure.) Make a third view, showing the edge view of the given plane (Art. 114). Then proceed as in Prob. 8, Art. 44. We may note, as a check, that the projections of the shortest distance will be perpendicular to the traces of the plane (Art. 116).

118. *The Projection of a Point or Line on a Plane.* The projection of a point or line on a plane located by traces is readily accomplished by obtaining an edge view of the plane. The method does not differ essentially from that of Problem 9, Art. 44.

119. *The Angle between a Line and a Plane.* The true angle between a straight line and a plane can be found as follows. From any point of the given line, draw a line perpendicular to the plane. Find the angle between these two lines. The complement of this angle is equal to the angle required. The angle between the lines is most readily found by revolution (Art. 107).

Problem 14 (repeated). To find the angle between a line and a plane.

See Fig. 148. Let the plane be a general oblique plane, given by its traces, *HQ* and *VQ*. Let *ab* be the given line. Through *a* draw the line *ac* perpendicular to plane *Q* (Art. 116, Cor. 2). Find the angle at *a* between lines *ab* and *ac*, here accomplished by revolving about the line *X* as axis (Art. 107, first example). This angle appears at a_r. Its complement is the true angle between line *ab* and the plane *Q*.

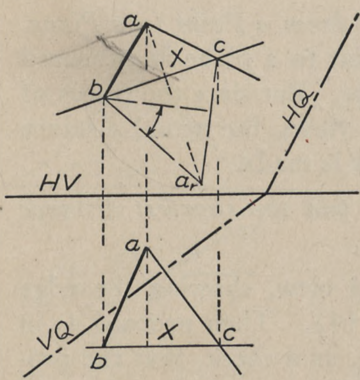

Fig. 148.

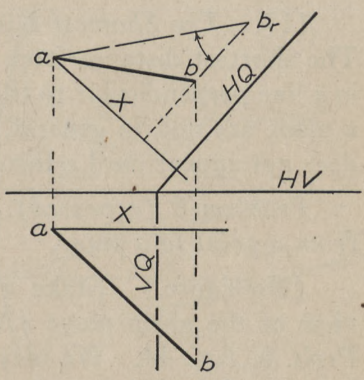

Fig. 149.

Let the plane be given so that one of its traces is also an edge view, as for example, *HQ* in Fig. 149. Then the line *X*, perpendicular to the plane from point *a*, shows in true length in the plan, and may be used as an axis of revolution (Art. 107, second example). If so used, the revolving point, *b*, moves in a plane parallel to *Q*. Hence the true angle between the line and plane is the same as that between the revolved position, ab_r, of the line and the edge view, *HQ*, of the plane.

In Fig. 150 the given plane is parallel to *HV*. In this situation, the line from point *a* perpendicular to plane *Q*, although a perfectly definite line in space, is not defined by its plan and elevation. A third view is needed. Take *RL2* so as to obtain an edge view of *Q*; project also the given line *ab*. Then the third and second views of Fig. 150 correspond respectively to the plan and elevation of Fig.

149. Solving in these views, the required angle is shown at b_r.

Note. This is obviously a shorter method of finding the angle between a line and a plane than that given in Art. 45. If desired to apply this solution to a plane given by a location other than traces, we may find the traces of the plane on any given or assumed planes of reference.

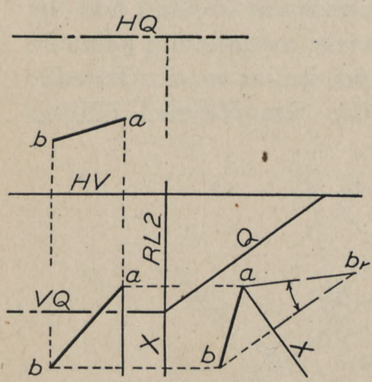

Fig. 150.

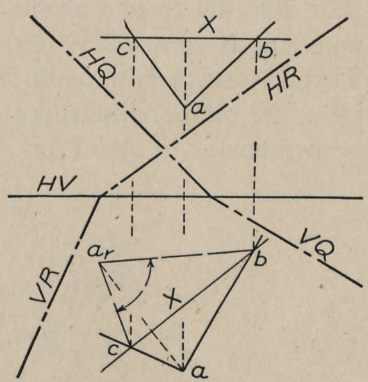

Fig. 151.

120. *The Angle between Two Planes.* If two planes are given by their traces, the angle between the planes may be variously found.

First Method. From any point in space, draw a line perpendicular to each of the given planes. The angle between the lines is equal to the angle between the planes.

Second Method. Pass a secant plane perpendicular to both given planes, intersecting each in a line. Find, by revolution or projection, the angle between these lines. It is the required angle.

Both of these methods will be shown.

Problem 4 (repeated). To find the true angle between two planes.

First Method. See Fig. 151. The given planes are Q and R, located by their traces. The plan and elevation

of a point, a, are assumed. Line ab is drawn perpendicular to Q, line ac is perpendicular to R (Art. 116). The true angle at a is found by revolution about the axis X (Art. 107). This angle is the same as that between the given planes.

Second Method. See Fig. 152. The given planes are Q and R, intersecting in the line ab (Art. 115). Take $RL2$ parallel to one of the projections of ab, and find the true length view a^3b^3. On this view, assume any point e^3. Through e draw the plane X perpendicular to ab. Its edge view, X^3, is perpendicular to a^3b^3. Its H-trace, HX, is perpendicular to a^hb^h (Art. 116).

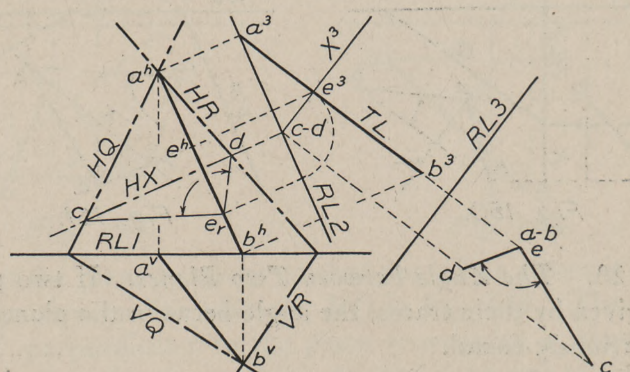

Fig. 152.

The point e is common to the three planes Q, R, and X. The plan shows that point c is common to Q and X (Art. 115), and that d is common to R and X. Therefore the plane X intersects Q and R in the lines ce and de, and the angle at e is required.

To obtain the angle by projection, take $RL3$ parallel to X^3, and project. Since $RL3$ is perpendicular to a^3b^3, this gives the end view of this line, and consequently the edge views of the given planes, as in Art. 43.

To obtain the angle by revolution, take the trace, HX, as the axis. Points c and d on this trace remain fixed; the

revolved position of e must be found. From the position of e^h on a^hb^h, and the fact that a^hb^h is perpendicular to HX, it is evident that e_r is on a^hb^h. Also, since $RL2$ is parallel to a^hb^h, the actual path of revolution of point e will project in the third view as shown. The required angle is $c{\cdot}e_r{\cdot}d$. Note that it is not necessary to find the projection e^h, once the manner of finding the revolved position e_r is understood.

121. *The True Size of a Plane Figure.* If the true size of a plane figure is found by revolution (Art. 107), a convenient axis is one trace of the plane containing the figure. This is shown in Fig. 153, where HQ is the axis.

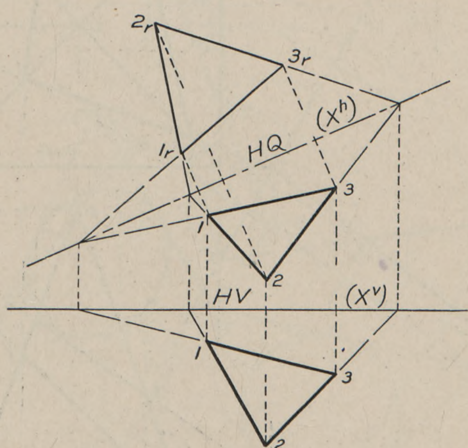

Fig. 153.

122. *Traces on a Specially Chosen Reference Plane.* The reference planes on which the plan and elevation of an object are drawn are, respectively, above and in front of the object (Art. 14). But the traces of oblique planes may, at times, be more conveniently drawn on other than these particular reference planes. For example, if the object rests on a horizontal plane as a base, that plane may be taken as one on which to draw traces. This is shown in Fig. 154. The pyramid rests on the horizontal plane H, shown edgewise in the elevation as H^v. The edges of the

base in the plan become, therefore, the traces of the sloping faces of the pyramid.

Since one line does not determine a plane, the traces *HR, HS,* and *HT* are not, in themselves, sufficient to locate the planes. But each plane contains, in addition, the point *o,* and is therefore completely determined, without drawing an additional trace.

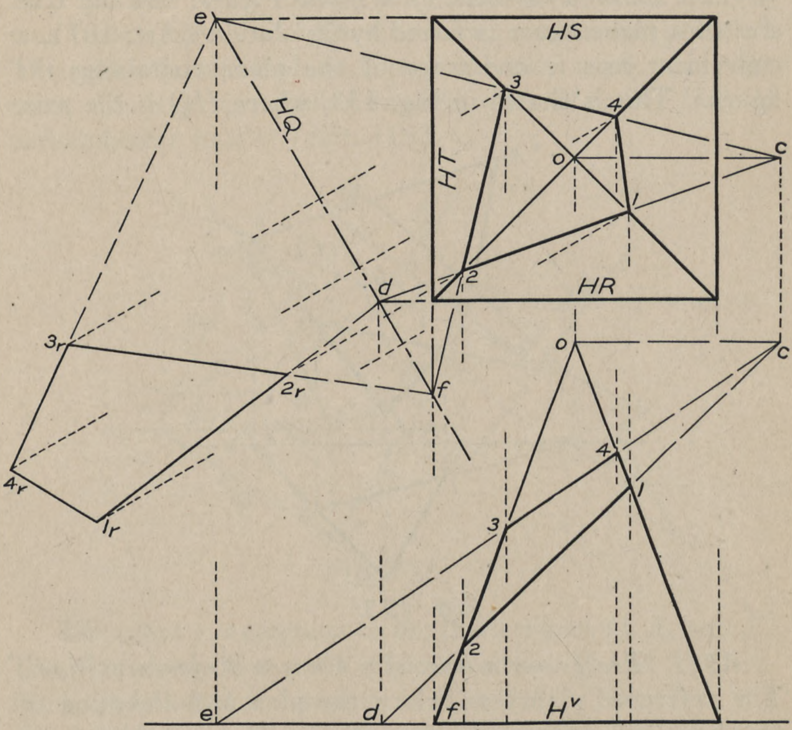

Fig. 154.

123. *An Oblique Section of a Pyramid.* Several problems have been solved in Fig. 154. Since planes *R* and *S* both contain point *o,* the line of intersection of these planes must be *oc.* Point *c* on this line being chosen as shown, through this point let us pass the plane *Q,* whose trace is *HQ.* Since point *c* is in the plane *R,* the intersection of

planes Q and R is cd. Similarly, since c is in plane S, the intersection of planes Q and S is ce. This enables us to construct the section *1·2·3·4*, made by plane Q. As a check, the intersection of planes Q and T passes through point f. Finally, the true size of the section has been found by revolution about HQ as an axis.

124. *Counter-Revolution.* The process of finding, by revolution, the true shape or size of a plane figure may be reversed, and the plan and elevation of a plane figure found from its true size.

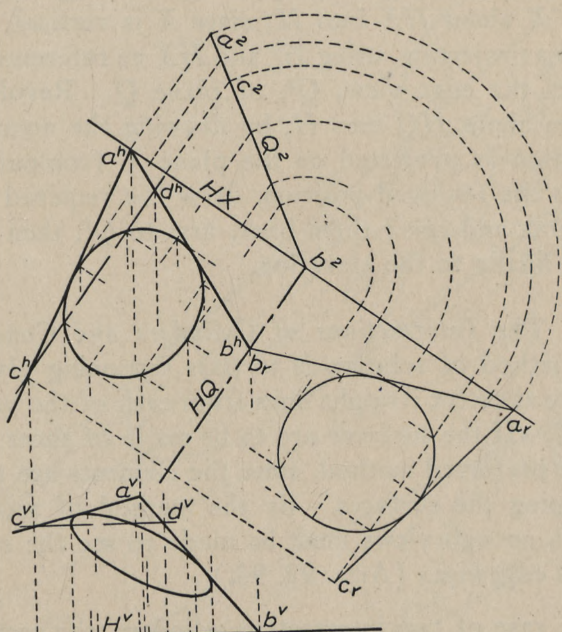

Fig. 155

Problem 13 (repeated). To construct a figure of a given shape and size in a given plane.

See Fig. 155. The figure to be constructed is a circle of a given radius tangent to the two lines, ab and ac, which locate the given plane. This problem may arise in locating a guide pulley, or in connecting two pipes by a circular bend.

Since measurements in such cases are usually taken upward from the floor, the horizontal reference plane is here taken below the lines, and is shown edgewise in the elevation as H^v (Art. 122).

Let Q be the plane of lines ab and ac. One point of its H-trace, HQ, is b^h. Since the line ac does not intersect H within reach, draw the horizontal line cd; then HQ is parallel to $c^h a^h$. Through point a pass the vertical plane X perpendicular to plane Q; that is, draw the H-trace (and edge view) HX through a^h perpendicular to HQ. Revolve the plane X about HX into H; since X is vertical, this is the same as projecting, using H^v and HX as reference lines. This gives the edge view, Q^2, of plane Q. Revolve the given lines about HQ into H, by drawing the actual arcs of revolution as projected on the plane X (compare Fig. 125). In the revolved position, draw the required circle. Select points and carry them back, first to Q^2, then to the plan, and finally to the elevation.

125. *The Intersections of Cylinders and Cones.* A general method of solution is to pass sectioning planes in such a way as to cut straight lines from each of the surfaces (Art. 92). If the surfaces are to be made of sheet metal, this is the preferred method, since the elements are needed in developing the surfaces. By the method of successive projection, enough views must be made to see the sectioning planes edgewise. (Arts. 92, 93.)

In the case of two intersecting cylinders, the sectioning planes are all parallel to each other. Therefore it is a simple matter to obtain a view in which all the sectioning planes appear edgewise simultaneously. Further, a cylindrical surface can itself be readily projected edgewise as a line (circle or ellipse). So that, when both the given surfaces are cylinders, there is usually so little need of using the traces of the sectioning planes that an example of this case will not be given.

But if one (or both) of the given surfaces is a cone, the case is different. The elements of a cone are not parallel, nor can the surface be projected orthographically as a single line.

126. *The Intersection of a Cone and a Cylinder.* The sectioning planes must contain the vertex of the cone, and at the same time be parallel to the axis of the cylinder. Hence every plane will contain a line, L, drawn through the vertex of the cone parallel to the axis of the cylinder.

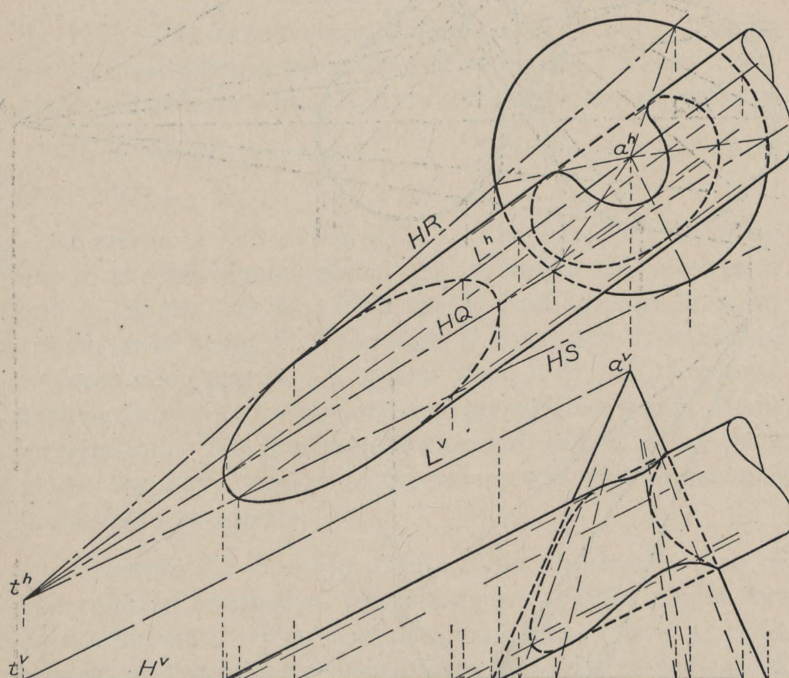

Fig. 156.

Problem 27 (repeated). To find the intersection of a cylinder and a cone.

See Fig. 156. The bases of the two surfaces lie in the same horizontal plane, H. Through the vertex, a, of the cone draw a line, L, parallel to the axis of the cylinder. Find the trace of this line on the plane H, the actual trace

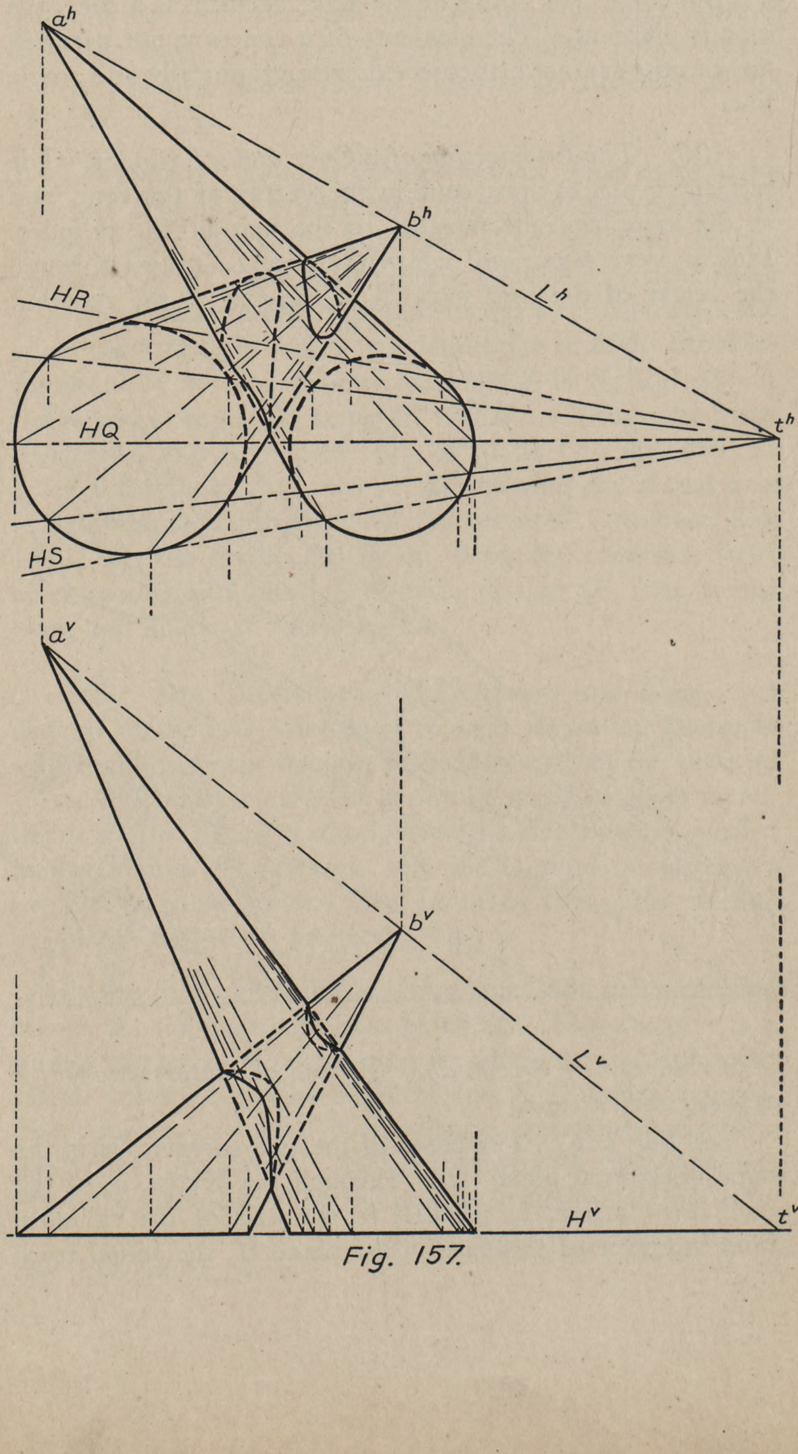

Fig. 157.

being t^h on L^h. Through t^h draw the H-traces of as many sectioning planes as may be necessary. A typical plane is Q, whose H-trace, HQ, intersects each base in two points. Each of these points locates an element of the surface, and the four elements thus determined give, by their intersections, four points in the curve.

The limiting planes (Art. 93) are readily seen to be R (HR) and S (HS), each of which is tangent to one of the bases and secant to the other.

127. *The Intersection of Two Cones.* The sectioning planes must contain the vertex of each cone. Hence every plane will contain a line, L, drawn through the two vertices.

Problem 28 (repeated). To find the intersection of two cones.

Example 1. See Fig. 157. The bases of the cones are in the horizontal plane, H. Connect the vertices, a and b, by the line L. Find the H-trace of this line, the actual trace being t^h. Through t^h draw the H-traces of the sectioning planes. A typical plane is Q (HQ), cutting each surface in two elements, and giving four points of the intersection. The limiting planes are R and S. Since these planes are both tangent to the same cone, the intersection breaks into two parts.

Example 2. See Fig. 158. This problem is that of two cones of revolution, whose axes are at right angles, but do not intersect. The bases are, consequently, also at right angles, and may conveniently be placed on a horizontal plane, H, and a profile plane, P, as shown by the elevation. Since the bases of the cones do not lie in a single plane of reference, as in Figs. 156 and 157, a single trace of each sectioning plane will not suffice.

Let L be the line joining the vertices a and b. Note in the elevation that this line pierces H at point t, and P at point u. Project to t^h and u^h in L^h.

Let *HP* be the trace of the plane *P* on *H*. Revolve *P* about *HP* into *H*, thus locating the trace *u* at u^p, and the center, *c*, of the base of one of the cones at c^p. Draw the circle representing this base.

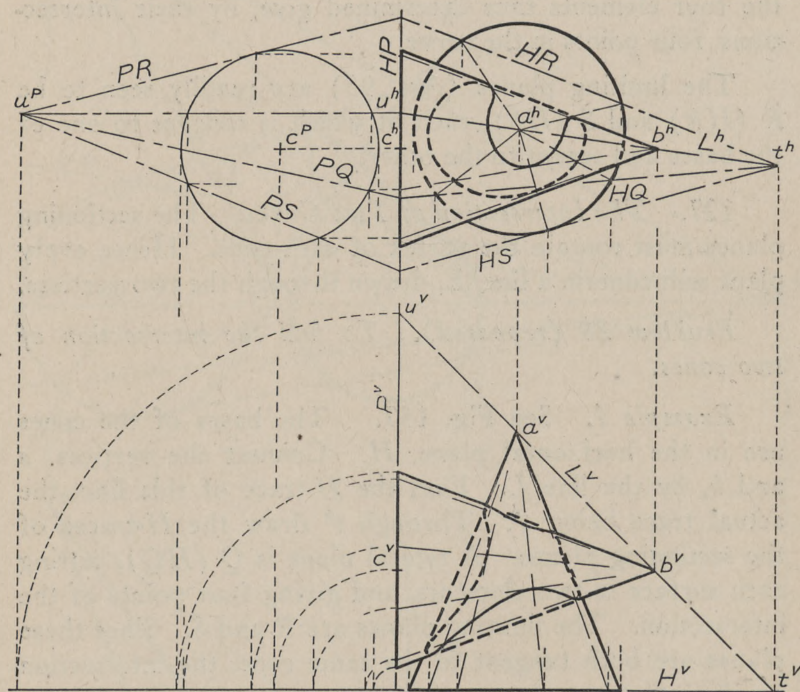

Fig. 158.

We are now ready to pass the sectioning planes. A typical plane is *Q*. The *H*-trace, *HQ*, passes through t^h; the *P*-trace, *PQ*, passes through u^p; *HQ* and *PQ* meet on *HP*. The intersections of *HQ* with the circular base determine two elements in one of the cones; the intersections of *PQ* with the circle determine two elements in the other cone. The limiting planes are *R (HR, PR)* and *S (HS, PS)*.

128. *Bases in Oblique Planes.* If the bases of the given surfaces do not lie in the same plane, nor in planes at right angles to each other, a solution by the method of

traces may sometimes be had by passing a reference plane so as to cut from each surface a section which may be used as a base. This is particularly the case if, as is usual in practical applications, both surfaces are surfaces of revolution, for one base can be made a circle and the other an ellipse whose axes are easily determined. See Fig. 156. In this figure, both the cone and cylinder are surfaces of revolution, the construction for obtaining the elliptical base of the cylinder being omitted.

129. *Conclusion.* It is not always possible to predict which method will give the most satisfactory solution to any particular problem that may arise. *The method of successive projection is based upon fundamentals, and will always lead to a solution (Art. 21). But, as has been shown, that method does not always give the shortest or nearest solution. So that acquaintance with some alternate method may occasionally prove of value.*

Chapter 10

Hip and Valley Angles for Roofs of Buildings

130. *The Dihedral Angle Between Two Planes.*

In the preparation of the working drawings for structures it is often necessary that the engineer know the true angle that one plane makes with another, and the angle that the line of intersection of the planes make with an edge

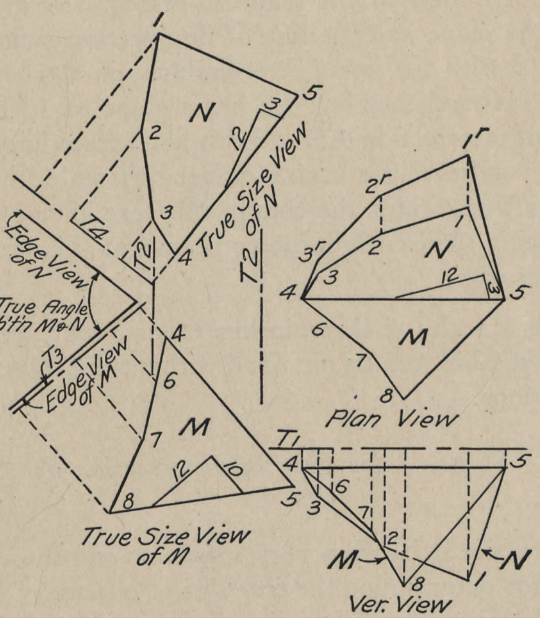

Fig. 159

of one or both planes. The details of the cutting of beams, and plates as well as the punching of holes and bending plates to connect the various beams in roof and other skew connections is explained in courses in structural design.

In Fig. 159 are shown two intersecting planes, *N* (*1*,*2*,*3*,*4*, *5*) and *M* (*4*,*5*,*6*,*7*,*8*). To determine the angle

between the planes the line of intersection *4,5* is projected on a third view as a point, the two planes projecting as edge views. The angle between these edge views is the true angle between the plans.

131. *Slope of the Line of Intersection by Projection Method.*

It is equally important and necessary to know the angle or slope the line of intersection of the two planes makes with the side of the plane.

A fourth view of the plane *N* in Fig. 159 gives a true size of the plane and the line of intersection makes a slope of *3* in *12* with the line *1,5*. Similarly on a true size view of *M* it is found that line *5,8* has a slope of *10* in *12* with the line of intersection *4,5*. In this and succeeding problems the slope or bevel has been measured by scale, the original drawings from which the cuts in this text have been reproduced, being sufficiently large to permit of this with reasonable accuracy.

With a scale of three inches to one foot, an accuracy of $\frac{1}{16}$ in. in 12 in. can be obtained without difficulty in graphical solutions.

132. *Slope of the Line of Intersection by Revolution Method.*

In practice it is often very simple to use the revolution method for determining the true size of a plane. The application of this method to finding the true size of plane *N* is shown in the plan view of Fig. 159, in which points *1,2,3* are revolved about *4,5* as an axis, to $1^r,2^r,3^r$. The plane *N* is now shown in its true shape and size by $4,5,1^r,2^r,3^r$ and the slope of *4,5* the line of intersection has a slope of *3* to *12* with the line *1,5* of the plane "*N*". Similarly the plane *M* could be revolved, about *4,5* as an axis, in either the plan view or the vertical view.

133. *Roof Planes.*

The plan and vertical views of the principal planes of roofs and similar structures can usually be shown as an edge in one view and in true or foreshortened size in the other view.

Fig. 160 represents planes as described above. Plane *1,2,3,4* is shown as an edge in the vertical view and a foreshortened view in plan, while the plane *5,6,7,8* is shown as an edge in the plan view and a foreshortened view in the vertical view.

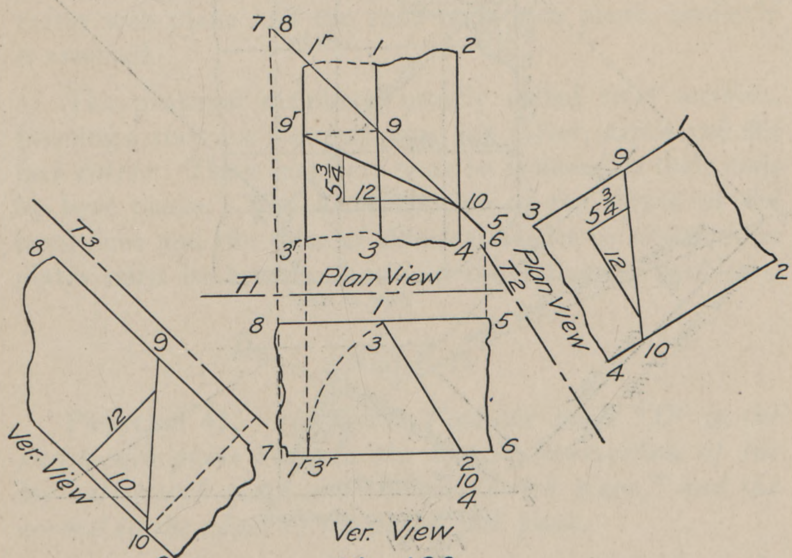

Fig. 160

The bevel made, by the line of intersection *9,10* on the plane *1,2,3,4* is *5-3/4* in *12* with its edge, while the bevel of this same line of intersection on plane *5,6,7,8* is *10* in *12* with its edge.

The dihedral angle or bevel between the two planes is not shown. This can be found by a fourth view in which the line of intersection *9,10* projects as a point.

134. *Principal Views for the Design of Hip and Valley Connections in Roof Construction.*

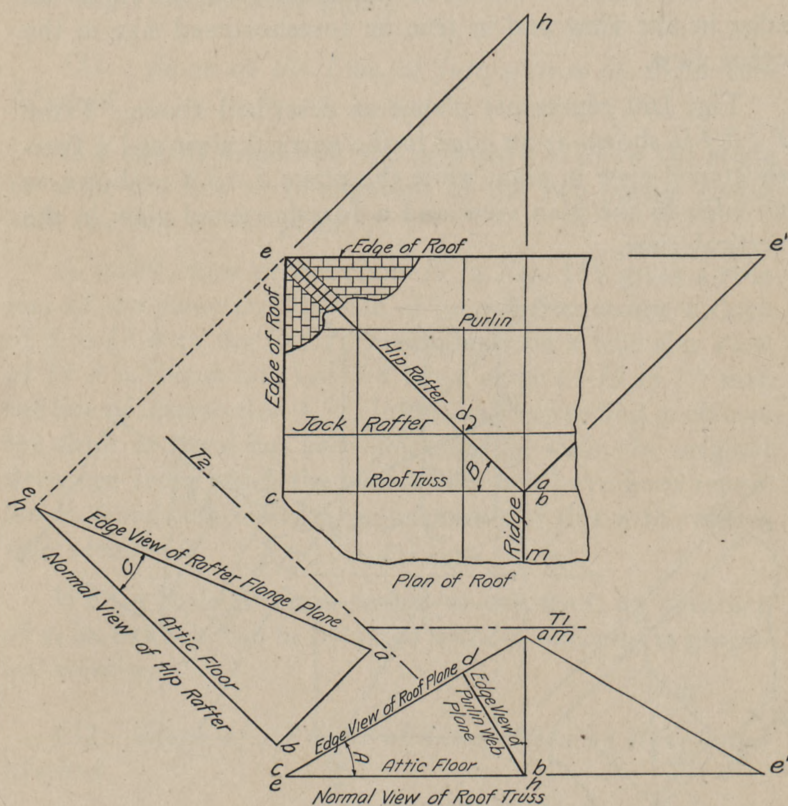

Fig. 161

The *plan of the roof,* the *normal view of the roof truss,* and the *normal view of the hip rafter,* shown in Fig. 161, are the *basic views* in the solution of hip and valley skew connections.

In the three views described above are represented the following principal planes of roof construction.

1.	Roof Truss Web Plane	*a-b-c*
2.	Roof Planes { side roof plane	*e-c-m-a*
	{ end roof plane	*e-e′-a*
3.	Hip Rafter Web Plane	*e-a-b*
4.	Attic Floor Plane	*b-h-e-c*
5.	Hip Rafter Flange Plane	*e-a-h*
6.	Edge View of Purlin Web Plane	*d-b*

7. Angle *"A"*, the pitch of the roof is assumed, that is, chosen so as to be satisfactory architecturally.

8. Angle *"B"*, the dihedral angle between the hip rafter web plane and the roof truss web plane, similarly is assumed.

The principal members, usually rolled steel sections, forming a roof are the *roof truss, hip rafter, purlin* and the *jack rafter*. These members must be fastened at their ends by bent plates. The determination of the slopes of the bend lines and the sizes of dihedral angles to which these plates must be bent is shown in the following problems.

ANGLE D

PROBLEM 41. *Fig. 162. Find the angle "D" in the purlin web plane between the line of intersection of the hip rafter web plane and the purlin web plane,* and the normal to the edges of the purlin web plane.*

In Fig. 162 the purlin web plane (*mnbhd*) is shown in edge view on the truss elevation, and the foreshortened view in plan, where (*bhd*) is an extension of the actual plane. The hip rafter web plane (*eab*) is shown as an edge in the plan view. The line of intersection of these two planes is *db*. The best method to obtain the true length of *"db"* on the purlin web plane is that of revolu-

* When a beam is cut off with a bevel the angle given on the drawing is that made by the required cut with the usual square cut.

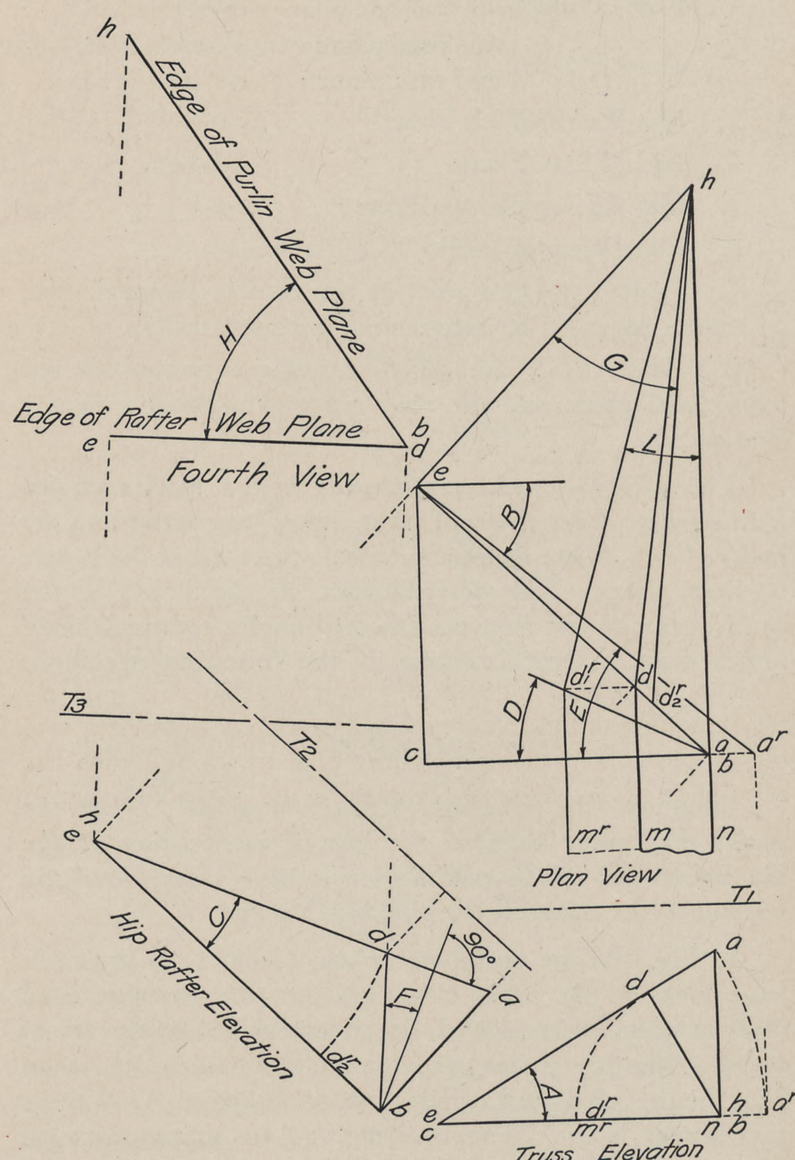
Edge of Purlin Web Plane
Edge of Rafter Web Plane
Fourth View
Plan View
Hip Rafter Elevation
Truss Elevation
90°
T1
T2
T3

Fig. 162

tion. Revolve the purlin web plane about the line *hb*, (the intersection of the attic floor plane and the purlin web plane) as an axis into the attic floor plane. In this revolution point *d* will move to d_1^r. Angle $d_1^r bc$, denoted as *"D"* is the required angle.

ANGLE L

PROBLEM 42. *Fig. 162. Determine the angle "L" in the purlin web plane between the line of intersection of the rafter flange plane and the purlin web plane, and an edge of the purlin web plane.*

In Fig. 162 the rafter flange plane (*aeh*) is shown as an edge view in the hip rafter elevation and a foreshortened view (represented in extension of the actual) on the plan view. (Remember that this plane intersects the attic plane in a line perpendicular to *eb* in plan).

The purlin web plane (*mnbhd*) has the same location as in problem 41.

The line of intersection of these two planes is *hd*. By the same method as in the previous problem the purlin web plane is revolved about the line *"hb"* as an axis into the attic floor plane, point *d* revolves to d_1^r, now $d^r h$ is a true length line and the angle $d_1^r hb$, denoted as *"L"*, is the required angle.

ANGLE G

PROBLEM 43. *Fig. 162. Determine the angle "G" in the rafter flange plane (eah) between the line of intersection of the purlin web plane (mnbhd) and rafter flange plane, and an edge of the rafter flange plane.*

As explained in problem 42 *"dh"* is the line of intersection of the two planes. In the plan view lines *ed* and *hd* do not show in true length, but *eh*, the line of intersection of the attic floor plane and the hip rafter flange plane is

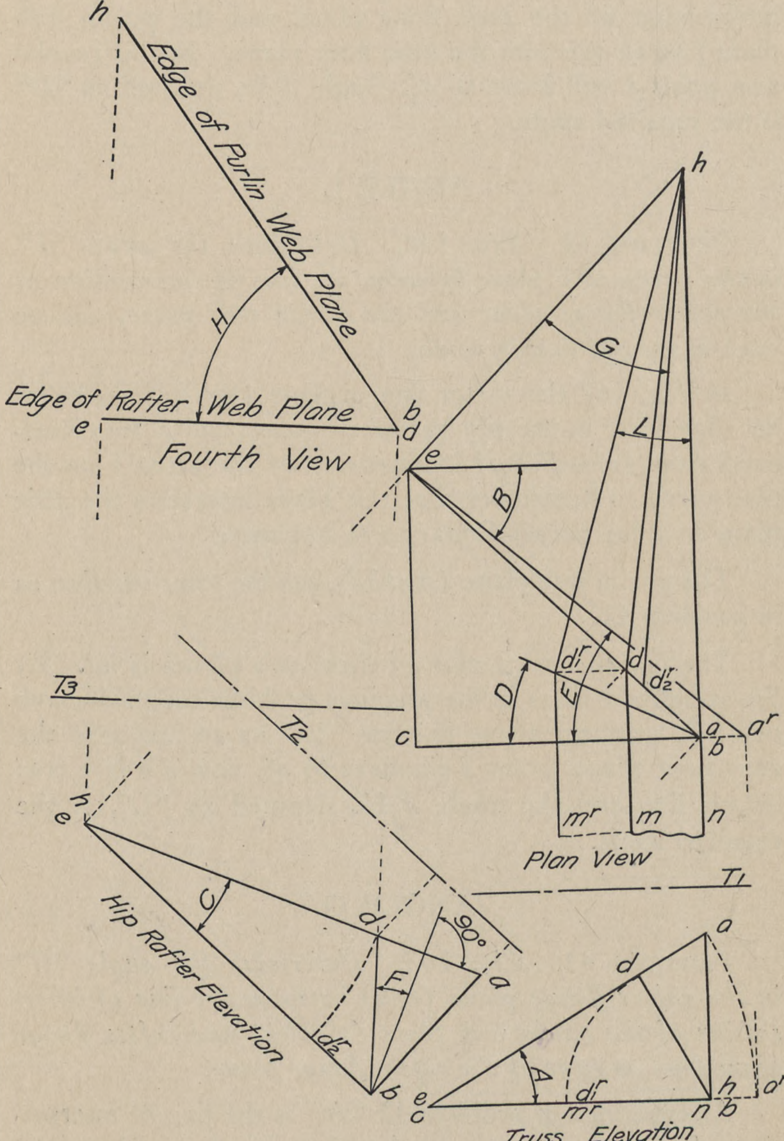
h
Edge of Purlin Web Plane
H
Edge of Rafter Web Plane
e
b
d
Fourth View
h
G
L
e
B
T3
T2
D
E
c
a
b
a^r
m^r
m
n
Plan View
T1
h
e
C
Hip Rafter Elevation
d
90°
F
a
b
a
d
A
e
c
h
a^r
n
b
Truss Elevation

Fig. 162

shown in true length. To determine the true angle, revolve the hip rafter flange plane into the attic floor plane about the line *eh* as an axis. Point *d* now takes the position of d^r_2 : d^r_2h now is a true length line and the angle ehd^r_2 denoted as *"G"*, is the required angle.

ANGLE E

PROBLEM 44. *Fig. 162. Determine the angle "E" in the roof plane between the intersection of the hip rafter web plane and the roof plane, and the normal (ac) to the edge (ec) of the roof plane.*

In Fig. 162 the roof plane is (*eca*) shown as an edge view on the truss elevation, and a foreshortened surface on the plan view. The hip rafter web plane is the same as in the previous problems.

Revolve the roof plane about *ec*, the intersection of roof plane and the attic floor plane, as an axis into the attic floor plane. Point *a* will revolve to a^r. The angle ca^re, denoted as *"E"*, is the required angle.

ANGLE F

PROBLEM 45. *Fig. 162. Determine the angle "F" in the hip rafter web plane between the line of intersection of the purlin web plane and the hip rafter web plane, and the normal to the edge ea* of the hip rafter web plane.*

To solve this problem project the line of intersection *"db"* on the "hip rafter elevation" which represents the true size of the hip rafter web plane. Accordingly *db* is the true length of the line of intersection. The angle between the line *db* and the perpendicular from *b* to the edge *ea* of the hip rafter flange plane, is the required angle *"F"*.

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* Line (*ea*) is parallel to the length of the beam forming hip rafter. See footnote 1, prob. 41.

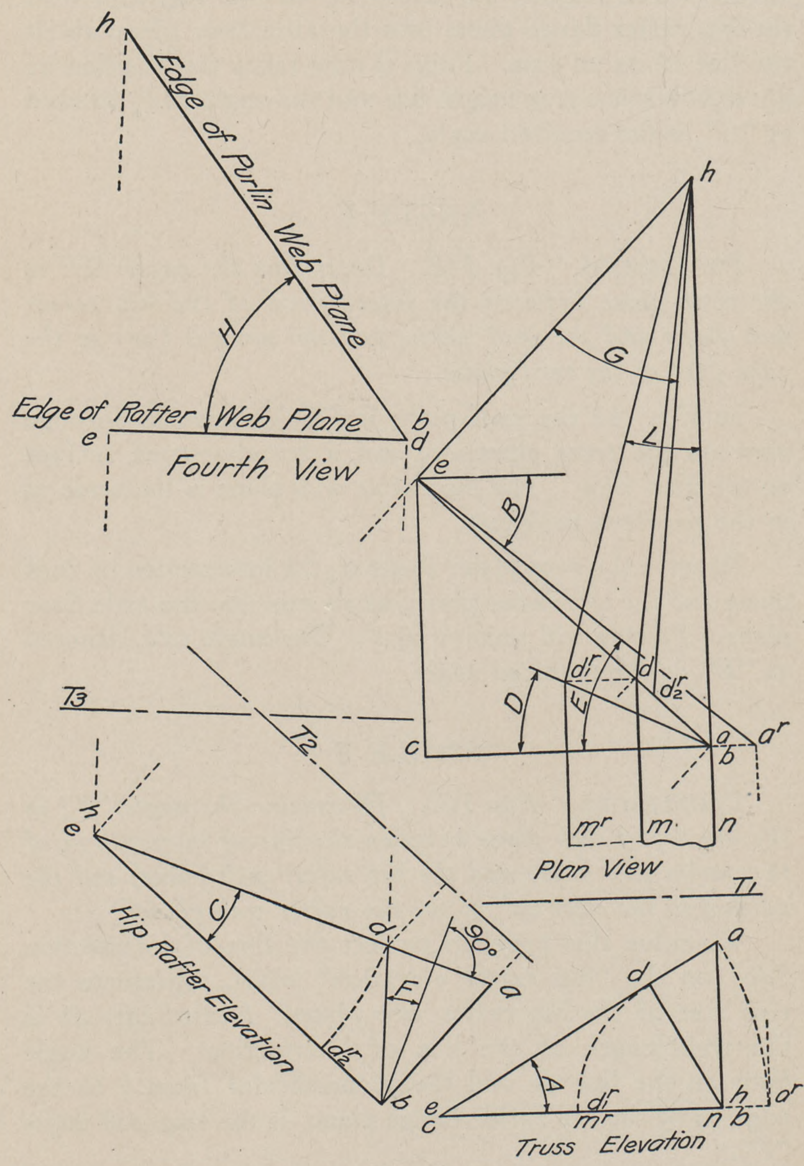
Edge of Purlin Web Plane
Edge of Rafter Web Plane
Fourth View
Plan View
Hip Rafter Elevation
Truss Elevation
T1
T2
T3
90°

Fig. 162

ANGLE H

PROBLEM 46. *Fig. 162. Determine the dihedral angle "H" between the hip rafter web plane and the purlin web plane.*

The true length of the line of intersection of the hip rafter web plane and the purlin web plane is *"db"* shown on the hip rafter elevation in Fig. 162. On a fourth view the line *db* projects as a point and the planes as edge views: the angle between these edge views is the true size of the angle *"H"*.

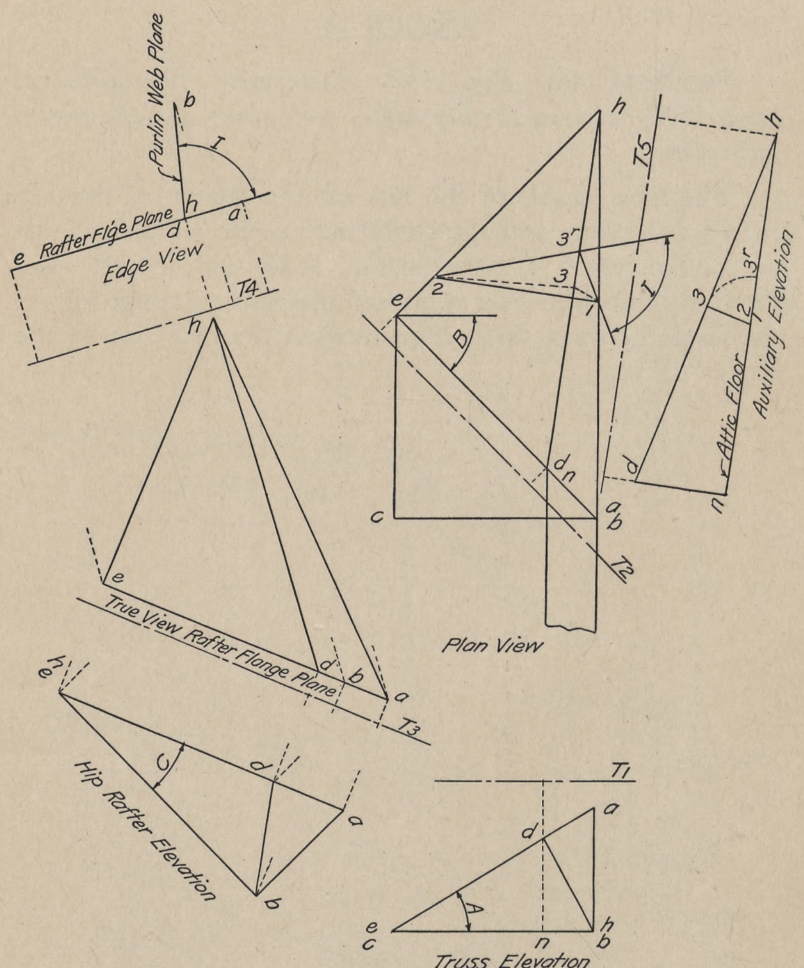

Fig. 163

ANGLE I

PROBLEM 47. *Fig. 163. Determine the dihedral angle I between the purlin web plane and the rafter flange plane.*

Solution No. 1.—The line of intersection between the two planes is "*dh*". Project the line of intersection on an

auxiliary plane parallel to the line *db*. On this plane the line *db* is shown in true length. At any point *3* on the line, pass a plane (*1,2,3*) perpendicular to the true length line *"dh"* on the auxiliary elevation. Revolve this plane about the line *1,2,* which is a line in the attic floor, point *3* moving to 3^r. The angle 23^r1, is the angle between the purlin web plane and the hip rafter flange plane, the supplement of this angle is represented by the angle *"I"* which is the angle used in the layout of hip and valley problems.

Solution No. 2.—In the "Edge View" of Fig. 163, the line of intersection *"dh"* of the two planes is projected as a point, and the two planes as edge views. The angle *"bha"* represents the angle *"I"* between the two planes.

ANGLE M

PROBLEM 48. *Fig. 164. Determine the dihedral angle "M" between the purlin web plane and a plane that is perpendicular to the hip rafter web and flange planes.* (Such is a normal or cross-sectional plane of the steel member forming the hip rafter).

Solution No. 1 (Method of Revolution)—In Fig. 164, the line *"ea"* is the line of intersection of the hip rafter flange plane (which is not shown otherwise) and the roof plane *eca*. The construction will require a true length view of this line of intersection: therefore revolve the roof plane into the attic floor plane about the line *"ec"* as an axis in this revolution *a* will revolve to a^r and the true length of the line of intersection is ea^r.

The edge view of the roof plane is the line *ea* and the edge view of the purlin web plane is the line *db,* both in the truss elevation. These planes are perpendicular to each other and therefore in the above revolution they remain perpendicular. The points *dab* take the position $d^ra^rb^r$, after revolution the roof plane coincides with the

attic floor plane, the purlin web plane becomes perpendicular to the attic floor in the truss elevation. Since ed^ra^r in the plan view is shown in true length, a plane passed through d^r in the plan view at 90° with the line is the edge

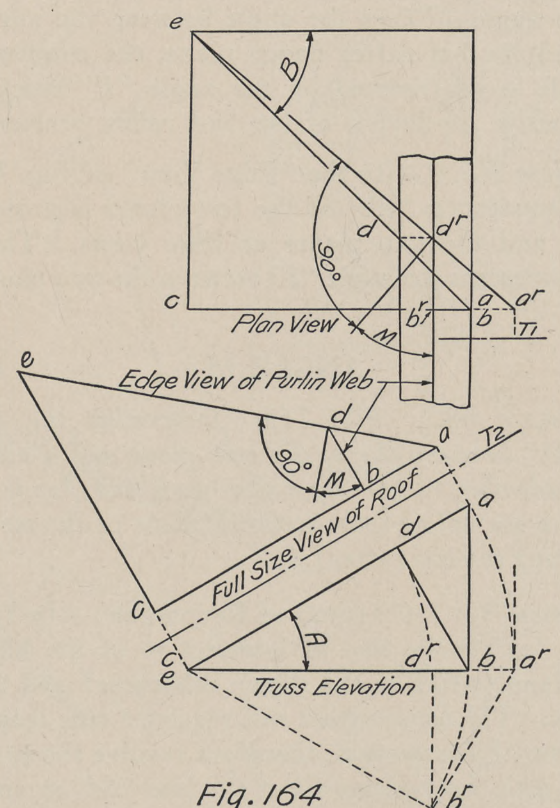

Fig. 164

view of a plane which is perpendicular both to the hip rafter flange plane and the hip rafter web plane. The dihedral angle between the two planes is represented by the angle *"M"* in the plan view.

Solution No. 2 (Method of Projection)—In the solution of this problem the true length of the line of intersec-

tion of the roof plane and the hip rafter web plane is required: since this is a line in the roof plane, a view showing a full size of the roof plane, is necessary. This is shown in a third view (that is, full size view of roof) of Fig. 164. The angle between the edge view of the purlin web plane and the edge view of a plane passed thru *d*, perpendicular to the true length line *"ea"*, is the required angle *"M"*.

ANGLE N

PROBLEM 49. *Fig. 165. Determine the angle in a plane perpendicular to both the hip rafter web plane and the hip rafter flange plane, lying between the line of intersection of this perpendicular plane and the purlin web plane, and a normal to edge of the perpendicular plane.*

The student is to write an explanation of the two solutions to this problem given in Fig. 165.

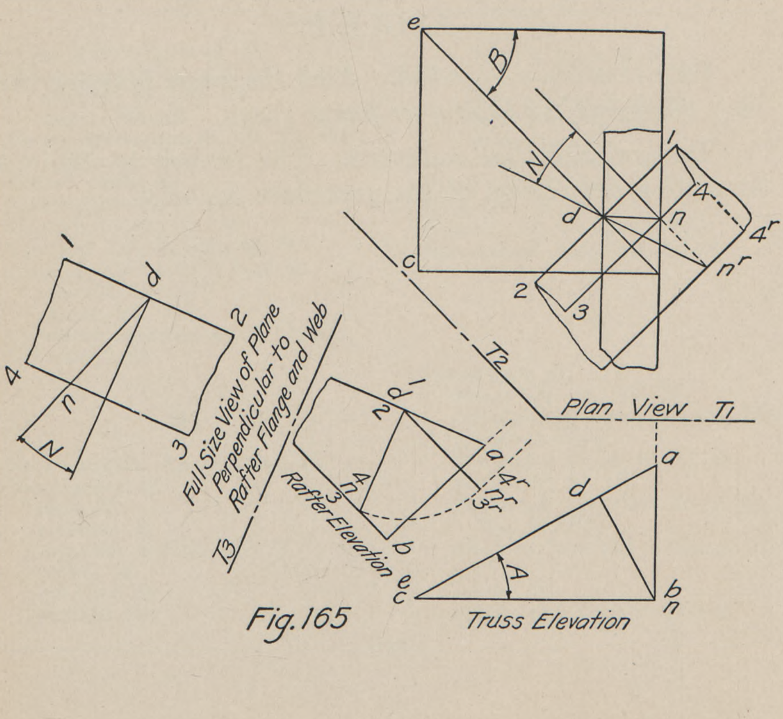

Fig. 165

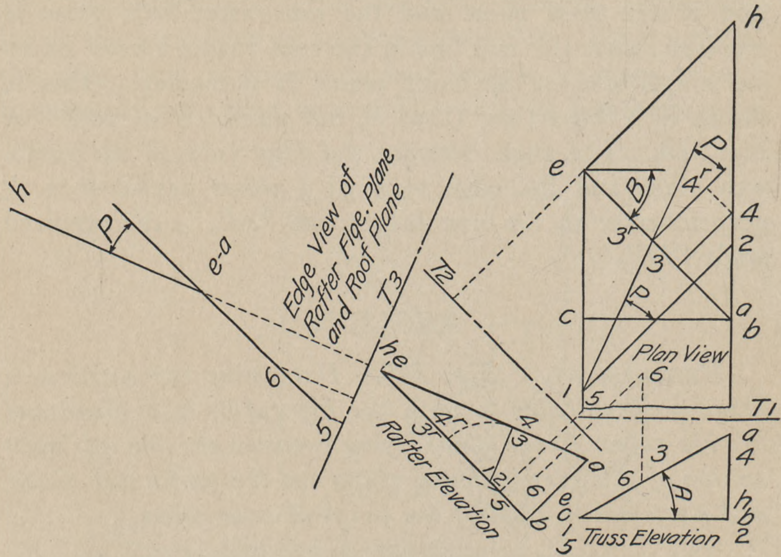

Fig. 166

ANGLE P

PROBLEM 50. *Fig. 166. Find the angle between the roof plane and the hip rafter flange plane.*

Two solutions are indicated. The student is required to write a description of the procedure followed.

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